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# GEOMETRICAL CONICS

BY

CHARLES SMITH, M.A.

MASTER OF SIDNEY SUSSEX COLLEGE, CAMBRIDGE.



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## PREFACE.

**I**N the first chapter of the following work I have proved some of the properties of conics by methods which are applicable to all the different forms, and have then proceeded to consider the three types of conics in detail in separate chapters. The book may, however, be said to have two beginnings; for the chapter on the Parabola is so written that it may be read first by those who prefer this arrangement.

In the first chapter I have made use of the Eccentric Circle of Boscovich to obtain a general proof of the fundamental theorem that the ratio of the rectangles contained by the segments of two chords of a conic drawn in fixed directions is constant. This method was first introduced into English text-books by Dr C. Taylor in his very able and comprehensive Treatise on the Ancient and Modern Geometry of Conics.

I have discarded the usual method of treating a hyperbola as if it were *two* conics, namely, the curve itself and

the conjugate hyperbola. This will, I hope, meet with the approval of teachers.

Numerous easy examples are given under the different propositions throughout the book; the examples at the end of each chapter are more difficult, and should not be attempted on the first reading.

The important properties of confocal conics are discussed at length, and complete solutions of other interesting and important theorems and constructions are given.

I have endeavoured to make the book suitable for beginners, but have included chapters on Cross-ratios, Reciprocation and Conical Projection, which are now included in the first year's course at Cambridge.

CHARLES SMITH.

*October, 1894.*

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## CHAPTER I.

**1. Definitions.** A **Conic Section**, or **Conic**, is the curve traced out by a point which moves in the plane containing a fixed point and a fixed straight line, in such a manner that its distance from the fixed point is in a constant ratio to its perpendicular distance from the fixed straight line.

The fixed point is called a **focus**, the fixed straight line is called a **directrix**, and the constant ratio is called the **eccentricity** of the conic.

It will be shewn hereafter that if a right circular cone be cut by any plane, the section is in all cases a conic as defined above. It was as sections of a cone that the properties of these curves were first investigated.

A conic is called an **ellipse**, a **parabola**, or a **hyperbola** according as its eccentricity is less than, equal to, or greater than unity.

**2.** Our object is to investigate the principal geometrical properties of conics. We first find the position and shape of the different curves.

**Prop. I.** *To find the points in which a conic, whose focus, directrix and eccentricity are given, is cut by the straight line through its focus perpendicular to the directrix.*

Let  $S$  be the focus and  $KK'$  the directrix of a conic.

Draw through  $S$  the line  $ZSZ'$  perpendicular to the directrix and cutting it in the point  $X$ .

In  $SX$  take a point  $A$  such that the ratio of  $SA$  to  $AX$  may be equal to the eccentricity of the conic; then  $A$  will be a point on the conic.

Divide  $XS$  externally in  $A'$  such that

$$SA' : A'X = SA : AX;$$

then  $A'$  will also be a point on the conic.

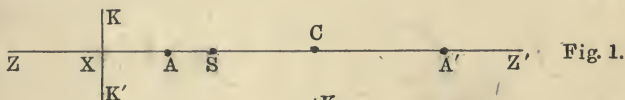


Fig. 1.



Fig. 2.

The point  $A'$  will be in  $XS$  produced if  $SA < AX$ , that is if the conic is an *ellipse* (Fig. 1); and the point  $A'$  will be in  $SX$  produced if  $SA > AX$ , that is if the conic is a *hyperbola* (Fig. 2).

If, in either case, we suppose the eccentricity of the conic to become more and more nearly equal to unity, the distance of the point  $A'$  from the focus will increase without limit. Hence one of the points in which the line  $ZZ'$  cuts the *parabola* whose focus is  $S$  and directrix  $KK'$  is at an *infinite distance* from  $S$ .

Thus the line through the focus of a conic perpendicular to its directrix cuts the conic in two points which are on the same side or on opposite sides of the directrix according as the conic is an *ellipse* or a *hyperbola*. Also the line through the focus of a *parabola* perpendicular to its directrix will only meet the curve in one point at a finite distance from the focus.

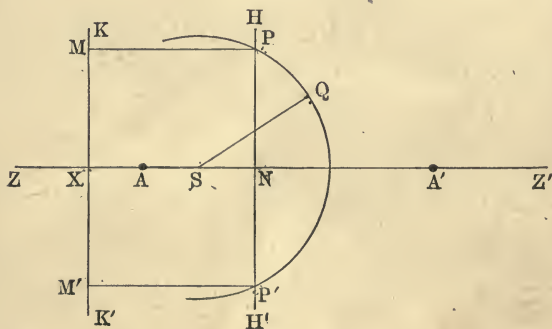
**3. Prop. II.** *To find the points in which a conic, whose focus, directrix and eccentricity are given, is cut by a straight line parallel to its directrix.*

Let  $S$  be the focus and  $KK'$  the directrix of the conic.

Draw through  $S$  the line  $ZSZ'$  perpendicular to the directrix and cutting it in the point  $X$ .

Through any point  $N$  in  $ZSZ'$  draw the line  $HNH'$  parallel to the directrix.

With centre  $S$  and radius  $SQ$  such that the ratio of  $SQ$  to  $NX$  is equal to the eccentricity, describe a circle cutting  $HH'$  in the points  $P, P'$ ; then  $P, P'$  will be points on the curve.



For, if  $PM, P'M'$  be perpendiculars on the directrix,

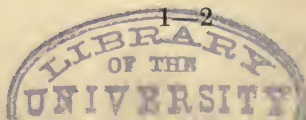
$$PM = NX = P'M'.$$

Hence  $SP : PM = SP' : P'M' = SQ : NX$ .

It will be proved in Article 6 that the circle described as above will intersect the line  $HNH'$  provided that  $N$  is between  $A$  and  $A'$  in the case of an ellipse, and is not between  $A$  and  $A'$  in the case of a hyperbola.

Since  $SP = SP'$ , and  $XSN$  is perpendicular to  $PP'$ ,  $PN$  must be equal to  $NP'$ .

Now when a straight line is so related to a curve that corresponding to any point of the curve there is another point such that the chord joining the two points



is bisected perpendicularly by that straight line, then the curve is said to be **symmetrical** about the straight line, and the straight line is called an **axis** of the curve.

Thus a **conic is symmetrical about the straight line through its focus perpendicular to its directrix**, which line is accordingly called an *axis* of the curve.

A point where an axis cuts the curve is called a **vertex**.

Thus the points  $A, A'$  in Art. 2 are vertices of the conic.

Sometimes the line  $AA'$ , terminated by the vertices, is called *the axis*.

4. If, in Article 2,  $C$  be the middle point of  $AA'$ , we have

$$\begin{aligned} SA : AX &= SA' : A'X; \\ \therefore SA : AX &= SA + SA' : AX + A'X \\ &= SA' - SA : A'X - AX. \end{aligned}$$

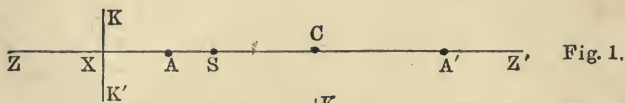


Fig. 1.

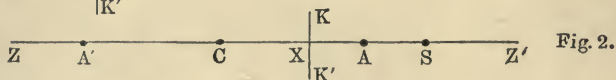


Fig. 2.

Hence in the ellipse (Fig. 1)

$$SA : AX = 2CA : 2CX = 2CS : 2CA;$$

and in the hyperbola (Fig. 2)

$$SA : AX = 2CS : 2CA = 2CA : 2CX.$$

Thus in both curves we have

$$CS : CA = CA : CX = SA : AX,$$

whence also

$$CA^2 = CS \cdot CX,$$

and

$$\begin{aligned} CS : CX &= CS^2 : CA^2 \\ &= SA^2 : AX^2. \end{aligned}$$

If the eccentricity of the conic be equal to the ratio  $e : 1$ , so that  $SA : AX = e : 1$ , and therefore  $SA = e \cdot AX$ , the above relations may be expressed more shortly thus

$$CS = e \cdot CA, \quad CA = e \cdot CX, \quad CS \cdot CX = CA^2,$$

and

$$CS = e^2 \cdot CX.$$

Ex. 1. If two points on a conic be equally distant from a focus, the line joining the points must be parallel to the directrix, and the focal distances of the points must be equally inclined to the axis.

Ex. 2. The directrix of a conic is given and also two points on the curve; shew that the focus must lie on a fixed circle.

Ex. 3. Find the focus of a conic whose directrix is given and also three points on the curve. How many conics can be drawn to satisfy the given conditions?

Ex. 4. A circle passes through a fixed point and cuts a fixed straight line at a given angle; shew that the centre of the circle must lie on a fixed hyperbola.

Ex. 5.  $S$  is the focus of a conic and  $P$  is any point on the curve. Shew that the locus of the middle point of  $SP$  is a conic of the same eccentricity whose focus is  $S$  and whose corresponding directrix is midway between  $S$  and the directrix of the original conic.

Ex. 6.  $S$  is the focus of a conic and  $P$  is any point on the curve; find the locus of the point  $Q$  which divides  $SP$  so that the ratio  $SQ : SP$  is constant.

Ex. 7. Shew that two conics with the same focus and directrix cannot intersect.

Ex. 8. Determine the directrix of a conic having given the focus, the eccentricity and two points on the curve. How many positions of the directrix can there be?

Ex. 9. Find the focus of a conic having given the directrix, the eccentricity and two points on the curve. How many possible positions of the focus can there be?

Ex. 10. Shew that the length of a focal chord of a conic is to twice the distance of its middle point from the directrix in the ratios of the eccentricity.

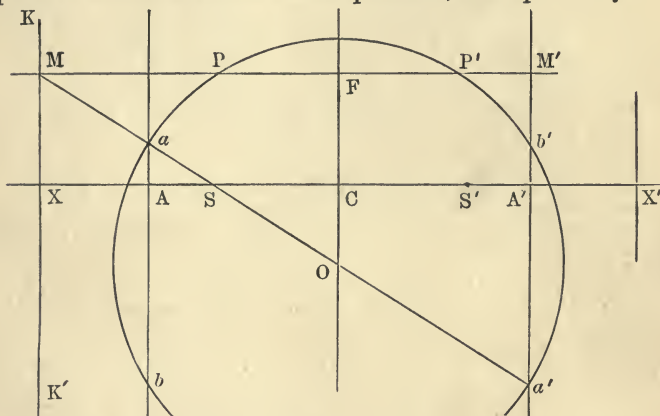
**5. Prop. III.** *To find the points in which a conic, whose focus, directrix and eccentricity are given, is cut by any line parallel to its axis.*

Let  $S$  be the focus and  $KK'$  the directrix.

Find  $A, A'$  the vertices of the conic, and let  $C$  be the middle point of  $AA'$ .

Let  $MM'$  be a line parallel to the axis cutting the directrix in  $M$ . We have to find the points of intersection of  $MM'$  and the conic.

Join  $MS$ , and let it meet the lines through  $A, A'$  parallel to the directrix in the points  $a, a'$  respectively.



Then, by similar triangles,

$$Sa : aM = SA : AX;$$

and

$$Sa' : a'M = SA' : A'X = SA : AX;$$

$$\therefore Sa : aM = Sa' : a'M.$$

Hence\*, if we describe a circle on  $aa'$  as diameter, and  $Q$  be any point on this circle, then will

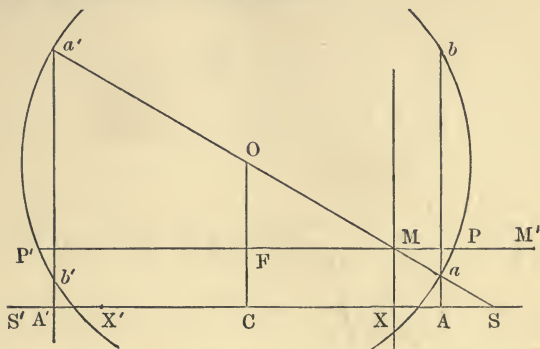
$$SQ : QM = Sa : aM = SA : AX.$$

If then the line  $MM'$  cut the circle in  $P, P'$  we shall have

$$SP : PM = SP' : P'M = SA : AX.$$

\* Taylor's *Euclid*, p. 426.

Hence, as  $P'PM$  is perpendicular to the directrix, the points  $P, P'$  must be on the conic.



Now, if  $O$  be the centre of the circle,  $O$  will be the middle point of  $aa'$ , and the line through  $O$  parallel to the directrix will therefore be equidistant from  $Aa$  and  $A'a'$ , and thus will pass through  $C$ , the middle point of  $AA'$ . But  $OC$  is perpendicular to  $PP'$  and therefore bisects  $PP'$ , in  $F$  suppose. Thus the middle point of  $PP'$  is on the line parallel to the directrix through the fixed point  $C$ .

Hence **the conic is symmetrical about the line through  $C$  parallel to the directrix**, which is therefore also an axis.

The axes perpendicular and parallel to the directrix are distinguished from one another by being called respectively the **transverse axis** and the **conjugate axis**.

6. If, in the preceding Article, the circle cut the lines  $Aa, A'a'$  again in the points  $b, b'$  respectively,  $ab'$  and  $ba'$  are both parallel to  $AA'$ , since the angles  $ab'a'$  and  $aba'$  are right angles.

In the case of the ellipse [Fig. 1],  $PP'$  is farther from the centre of the circle than  $ab'$  or  $a'b$ , since  $a$  and  $a'$  are on the same side of  $M$ . Hence  $PP' < ab'$ , that is  $PP' < AA'$ , whence it follows that *the ellipse lies entirely between the lines  $Aa$  and  $A'a'$* .

If  $P$  be any point on the ellipse and  $PM$  be the perpendicular on the directrix; then, since  $P$  is between the lines  $Aa$  and  $A'a'$ ,  $PM$  must be less than  $A'X$ , and therefore  $SP$  less than  $SA'$ . Thus every point of an ellipse is at a limited distance from the focus. Hence an ellipse is a *closed oval curve*.



If  $BCB'$  be parallel to the directrix, and  $B, B'$  be such that  $SB = SB' = e \cdot CX$ ; then  $B, B'$  are points on the ellipse, and are the extremities of the conjugate axis.

In the case of the hyperbola [Fig. 2],  $PP'$  is nearer to the centre of the circle than  $ab'$  or  $a'b$ , since  $M$  is between  $a$  and  $a'$ .

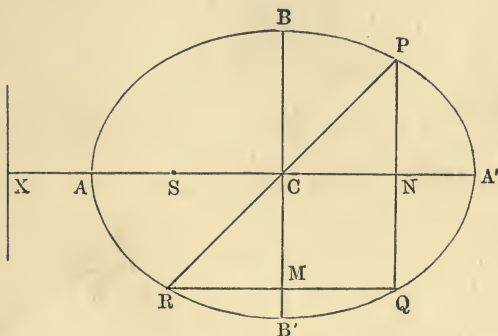
Hence  $PP' > AA'$ , whence it follows that *the hyperbola lies entirely without the lines  $Aa$  and  $A'a'$* . Since  $M$  is *within* the circle, the line  $MM'$  will always cut the circle in real points; moreover it is easily seen that  $PP'$  increases indefinitely as  $XM$  is increased.

Thus *the hyperbola is a curve with two distinct branches*, as in the figure.



**7. Central Conics.** Let  $P$  be any point on an ellipse or a hyperbola. Draw through  $P$  a line parallel to the directrix cutting  $AA'$  in  $N$  and the curve again in  $Q$ . Then  $PN = NQ$  [Prop. II.]. Now draw through  $Q$  a straight line parallel to  $AA'$  cutting the curve in  $R$  and the line through  $C$  parallel to the directrix in  $M$ ; then  $QM = MR$  [Prop. III.].

Since  $PQ = 2PN$ , and  $QR = 2QM = 2NC$ , it follows that  $PCR$  is a straight line, and that  $CR = PC$ .



Thus, if  $P$  be any point on an ellipse or a hyperbola, and  $PC$  be produced to  $R$  so that  $CR = PC$ ; then the point  $R$  will also be on the conic, so that all chords of the conic drawn through the point  $C$  are bisected at  $C$ . On this account the point  $C$  is called the **centre** of the conic.

The ellipse and hyperbola are called **central conics** to distinguish them from the parabola which has no centre, or rather whose centre is at an infinite distance from the focus.

**Note.** The parabola can be considered as the limiting form of an ellipse or of a hyperbola. It is a very instructive exercise to deduce from any property of an ellipse or hyperbola the corresponding property of the parabola, when the properties of the two curves are not precisely the same; this should however be deferred until the geometrical properties of the parabola have been considered in detail in the next chapter.

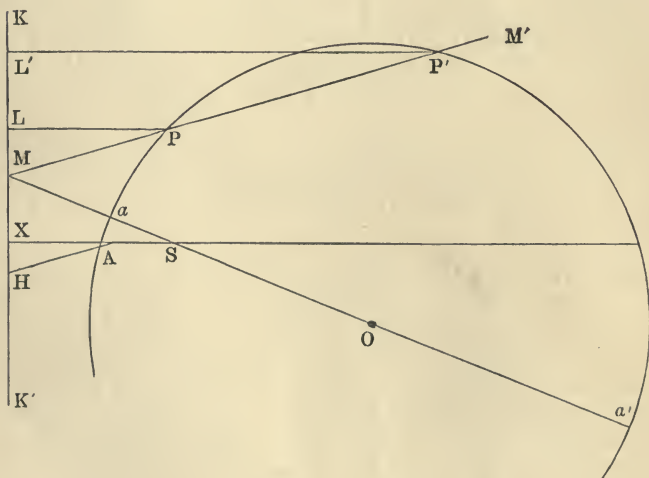
**8. Prop. IV.** *To shew that a central conic has two foci and two directrices.*

We first prove as in Art. 5 that an ellipse or hyperbola is symmetrical about the line through  $C$  parallel to the directrix. From this it follows that if the points  $S', X'$  be taken on the transverse axis such that  $CS' = SC$  and  $CX' = XC$ , the point  $S'$  will have the same properties with respect to the curve as the point  $S$  has; hence  $S'$  will also be a focus of the conic and the corresponding directrix will be the line through  $X'$  parallel to the original directrix.

Thus **an ellipse or a hyperbola has two foci and two directrices.**

**9. Prop. V.** *To find the points in which any given straight line cuts a conic whose focus, directrix and eccentricity are given.*

Let  $S$  be the focus and  $KXK'$  the directrix of the conic.



Let  $MM'$  be the given straight line cutting the directrix in  $M$ .

In  $SX$  take a point  $A$  such that  $SA : AX$  may be equal to the given eccentricity, and draw  $AH$  parallel to  $MM'$  cutting the directrix in  $H$ .

Join  $SM$  and divide it internally and externally at the points  $a, a'$  in the ratio  $SA : AH$ . Then, if we describe a circle on  $aa'$  as diameter, and  $Q$  be any point on this circle, the ratio of  $SQ$  to  $QM$  will be equal to the ratio of  $Sa : aM$ .

Let the circle cut the line  $MM'$  in the points  $P, P'$ ; then  $P, P'$  will be points on the conic.

Draw  $PL, P'L'$  perpendiculars on the directrix.

Then, since  $P$  is on the circle whose diameter is  $aa'$ ,

$$SP : PM = Sa : aM = SA : AH;$$

$$\therefore SP : SA = PM : AH.$$

But, from the similar triangles  $LPM$  and  $XAH$ ,

$$PM : AH = PL : AX.$$

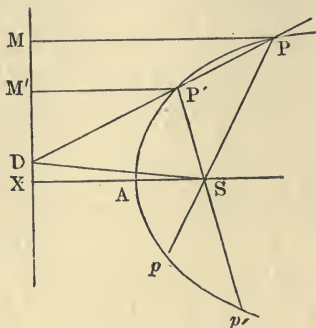
$$\text{Hence} \quad SP : SA = PL : AX,$$

whence it follows that  $P$ , and similarly  $P'$ , is a point on the conic.

It should be noticed that in the case of the ellipse or parabola the points  $a, a'$ , and therefore also the points  $P, P'$  are on the same side of  $M$ , that is on the same side of the directrix. In the case of the hyperbola however the points  $P, P'$  will or will not be on the same side of the directrix, according as  $SA$  is less or greater than  $AH$ ; and, although  $SA > AX$ , it does not follow that  $SA > AH$ .

It should also be noticed that if the direction of the chord be changed so that  $AH$  may become more and more nearly equal to  $SA$ , the distance  $Sa'$  will increase without limit, as will also the distance  $MP'$ ; and that when the direction is such that  $SH = SA$ , one of the two points in which  $MM'$  cuts the conic will be at an *infinite* distance from the directrix.

**10. Prop. VI.** *If a straight line cuts a conic, whose focus is  $S$ , in the points  $P, P'$  and the directrix in the point  $D$ ; then will  $SD$  be equally inclined to  $SP$  and  $SP'$ .*



Join  $SP, SP', SD$ , and draw the perpendiculars  $PM, P'M'$  on the directrix. Produce  $PS$  and  $P'S$  to meet the conic again in  $p, p'$  respectively. Then, by definition,

$$SP : PM = SP' : P'M' ;$$

$$\therefore SP : SP' = PM : P'M'.$$

But, from the similar triangles  $MPD, M'P'D$ , we have

$$PM : P'M' = PD : P'D.$$

Hence

$$SP : SP' = PD : P'D,$$

whence it follows that  $DS$  bisects the angle  $PSp'$  provided  $P, P'$  are both on the same side of  $D$ , and that  $DS$  bisects the angle  $PSP'$  provided  $P, P'$  are on opposite sides of the directrix, which last condition can, however, only be the case when the conic is a hyperbola.

**Cor. I.** A straight line can only cut a conic in two points.

For, if  $DPP'P''$  is a straight line, the points  $P, P', P''$  being on the conic, the line  $DS$  would have to make equal angles with  $SP, SP'$  and  $SP''$ ; and this is impossible.

**Cor. II.** If  $PSp$ ,  $P'Sp'$  be any two focal chords of a conic, the lines  $PP'$ ,  $pp'$  will meet on the directrix, as will also the lines  $Pp'$  and  $P'p$ .

For, if  $PP'$  meet the directrix in  $D$ ; then, as we have just proved,  $DS$  will bisect the angle  $PSp'$ ; and the line joining  $S$  to the point where  $pp'$  cuts the directrix will also bisect the angle  $P'Sp'$ . Hence  $PP'$  and  $pp'$  must cut the directrix in the same point.

Similarly  $Pp'$  and  $P'p$  will both meet the directrix in a point  $D'$  such that  $D'S$  bisects the angle  $psp'$ .

The lines  $DS$ ,  $D'S$  will clearly be at right angles to one another.

Ex. 1. Shew that, if the focus of a conic be given and also two points on the curve, the directrix must pass through one or other of two fixed points.

Ex. 2. Find the directrix of a conic having given a focus and three points on the curve; and shew that three at least, of the four possible conics, must be hyperbolas.

Ex. 3. Find the directrix of a conic having given a focus, the direction of the transverse axis, and two points on the curve.

Ex. 4.  $P$ ,  $P'$  are the extremities of a focal chord of a conic, and  $Q$  is any other point on the curve;  $PQ$ ,  $P'Q$  cut the corresponding directrix in  $K$ ,  $K'$  respectively. Shew that  $KK'$  subtends a right angle at the focus.

Ex. 5.  $PSP'$  is a focal chord, and  $A$  is a vertex of a conic;  $PA$ ,  $P'A$  cut the corresponding directrix in  $K$ ,  $K'$  respectively. Shew that

$$KX \cdot XK' = XS^2,$$

$X$  being the foot of the directrix.

Ex. 6.  $P$  is any point on a conic whose focus is  $S$ , and  $A$  is a vertex of the conic.  $PA$  cuts the corresponding directrix in  $K$ , and  $KQ$  is drawn parallel to the transverse axis to meet  $PS$  produced in  $Q$ . Shew that the locus of  $Q$  is a parabola. *major*

Ex. 7. Find the focus of a conic having given the directrix, a vertex and one other point on the curve.

Ex. 8. Find the directrix of a conic having given the focus, the vertex, and one other point on the curve.

Ex. 9. Shew that, if  $P$ ,  $P'$  be the two extremities of any central chord of a conic whose focus is  $S$ , then  $SP + SP'$  will be constant.

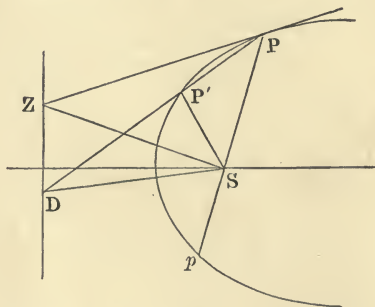
Ex. 10. Shew that, if two conics have a common directrix, their points of intersection must lie on a circle whose centre is on the line joining their foci.

**11. Definitions.** Let two neighbouring points  $P, P'$  be taken on any curve, and let the point  $P'$  move along the curve nearer and nearer to the point  $P$ ; then the limiting position of the line  $PP'$ , when  $P'$  moves up to and ultimately coincides with  $P$ , is called the **tangent** to the curve at the point  $P$ . Also the line through any point  $P$  of a curve perpendicular to the tangent thereat is called the **normal** to the curve at the point  $P$ .

**12. Prop. VII.** *The portion of a tangent to a conic intercepted between its point of contact and a directrix, subtends a right angle at the corresponding focus.*

Let the straight line  $DPP'$  cut a conic in the points  $P, P'$  and a directrix in the point  $D$ , the points  $P, P'$  being on the same side of the directrix. Then, if  $S$  be the focus corresponding to that directrix, and  $PS$  produced cut the conic again in  $p$ , it can be proved as in Art. 10 that  $DS$  bisects the angle  $P'Sp$ .

Now let  $P'$  move up to and ultimately coincide with  $P$ , and let  $PZ$  be the ultimate position of the line  $PP'$ , that is of the tangent at  $P$ . Then  $DS$  will always make



equal angles with  $P'S$  and  $Sp$ , and therefore ultimately, when  $P'$  has moved up to  $P$ , and  $D$  to  $Z$ ,  $ZS$  will make equal angles with  $PS$  and  $Sp$ . Hence each of the angles  $ZSP, ZSp$  is a right angle.

Thus  $ZP$  subtends a right angle at  $S$ .

Conversely, if  $SZ$  be drawn perpendicular to  $SP$  to meet the directrix in  $Z$ , then  $PZ$  will be the tangent at  $P$ .

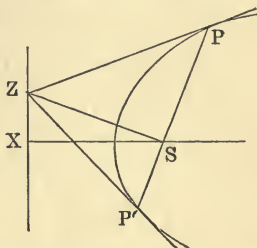
**13. Prop. VIII.** *Tangents at the extremities of a focal chord of a conic intersect in the corresponding directrix.*

Let  $PSP'$  be any focal chord of a conic whose focus is  $S$ . Draw  $SZ$  perpendicular to  $PSP'$  meeting the directrix corresponding to the focus  $S$  in  $Z$ .

Then, since the angles  $ZSP$ ,  $ZSP'$  are right angles,  $ZP$  and  $ZP'$  are tangents.

Thus the tangents at  $P$  and  $P'$  intersect on the directrix.

Conversely, if tangents be drawn to a conic from any point on a directrix, the line joining the points of contact will pass through the corresponding focus.



If  $PM$ ,  $P'M'$  be the perpendiculars on the directrix; then  $M$ ,  $Z$ ,  $S$ ,  $P$  are cyclic. Hence  $\angle SZP \cong \angle MZP$  according as  $SP \cong PM$ ;

so also  $\angle SZP' \cong \angle M'ZP'$  „ „ „ .

Hence  $\angle PZZ' \cong$  a right angle „ „ „ ,

[provided, however, that if the conic is a hyperbola the points  $P$ ,  $P'$  are the same side of the directrix].

**Def.** The focal chord of a conic perpendicular to the transverse axis is called the **Latus Rectum**.

From the above it is easily seen that the tangents at the extremities of the latus rectum intersect in the point  $X$ .

**14. Prop. IX.** *If from any point  $Q$  the perpendicular  $QM$  be drawn to the directrix of a conic, and  $S$  be the corresponding focus; then will the ratio  $SQ : QM$  be greater or less than the eccentricity according as the point is without or within the conic.*

A point  $Q$  is without a conic when the line  $SQ$  cuts the conic in one and only one point between  $S$  and  $Q$ .

Let  $Q$  be without the conic, and let  $SQ$  cut the conic in  $P$ . Draw  $PN$  perpendicular to the directrix, and join  $SM$ . Then  $SM$  will cut  $PN$  at a point  $K$  between  $P$  and  $N$ .

Since  $KP$  is parallel to  $QM$ ,

$$SQ : QM = SP : PK.$$

But  $SP : PK$  is greater than  $SP : PN$ ;

$\therefore SQ : QM$  is greater than  $SP : PN$ ,  
that is greater than the eccentricity of the conic.

If  $Q$  and  $S$  be on opposite sides of the directrix, and if  $SQ$  cut the conic in the two points  $P, P'$ , of which  $P$  is between  $S$  and  $Q$ . Then, if  $P'N'$  be drawn perpendicular to the directrix,  $SM$  produced will cut  $P'N'$  in a point  $K'$  between  $P'$  and  $N'$ ; whence it follows as above that  $SQ : QM$  is greater than  $SP' : P'N'$ .

It can be proved in a similar manner that if  $Q$  be within the conic,  $SQ : QM$  will be less than the eccentricity of the conic.

**15. Prop. X.** *If from any point  $T$  on the tangent at a point  $P$  of a conic,  $TH$  be drawn perpendicular to the directrix and  $TK$  perpendicular to the focal distance  $SP$ ; then will the ratio  $SK : TH$  be equal to the eccentricity.*

Let the tangent at  $P$  meet the directrix in  $Z$ . Draw  $PN$  perpendicular to the directrix.

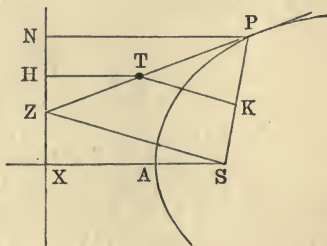
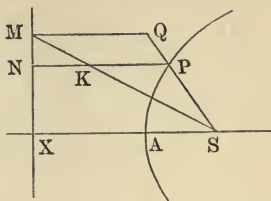
Then  $ZS$  is perpendicular to  $SP$ , and therefore parallel to  $TK$ .

Hence

$$\begin{aligned} SK : SP &= ZT : ZP \\ &= TH : PN. \end{aligned}$$

Hence

$$\begin{aligned} SK : TH &= SP : PN \\ &= SA : AX. \end{aligned}$$



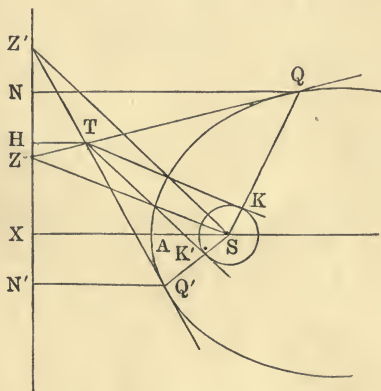
**16. Prop. XI.** *To draw tangents to a conic from an external point.*

Let  $T$  be the external point. Draw  $TH$  perpendicular to the directrix, and with  $S$  as centre describe a circle whose radius is to  $TH$  as  $SA : AX$ .

Since  $T$  is without the conic,  $ST : TH$  is greater than  $SA : AX$ . Hence  $ST$  is greater than the radius of the circle. We can therefore draw two real tangents,  $TK$ ,  $TK'$  suppose, to the circle.

Draw  $SZ$ ,  $SZ'$  parallel to  $TK$ ,  $TK'$  meeting the directrix in  $Z$ ,  $Z'$  respectively.

Join  $ZT$ ,  $Z'T$  and produce  $SK$ ,  $SK'$  to meet  $ZT$ ,  $Z'T$  respectively in  $Q$ ,  $Q'$ ; then  $TQ$ ,  $TQ'$  will be the tangents required.



Draw  $QN$ ,  $Q'N'$  perpendicular to the directrix.

Then, since  $SZ$  is parallel to  $TK$ ,

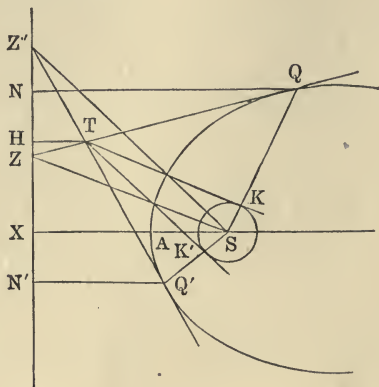
$$SQ : SK = ZQ : ZT = QN : TH;$$

$$\therefore SQ : QN = SK : TH$$

$$= SA : AX, \text{ by construction.}$$

Hence the point  $Q$  is on the conic, and therefore as  $ZSQ$  is a right angle,  $ZTQ$  is the tangent at  $Q$ . Similarly  $Z'TQ'$  is the tangent at  $Q'$ .

17. **Prop. XII.** *To shew that the two tangents drawn to a conic from an external point subtend equal or supplementary angles at a focus.*



Let  $TQ$ ,  $TQ'$  be the two tangents, and let  $TQ$ ,  $TQ'$ , produced if necessary, cut the directrix in  $Z$ ,  $Z'$  respectively.

Draw  $TH$  perpendicular to the directrix, and  $TK$ ,  $TK'$  perpendicular to  $SQ$ ,  $SQ'$  respectively.

$$\begin{aligned} \text{Then [Prop. XI.] } SK : TH &= SA : AX \\ &= SK' : TH. \end{aligned}$$

Hence

$$SK = SK',$$

whence it follows that  $T$  is on the internal bisector of the angle  $KSK'$  and therefore on the internal or on the external bisector of the angle  $QSQ'$ .

If the points  $Q$ ,  $Q'$  are on the same side of the directrix, and if  $T$  be also on that side of the directrix; then it is easily seen that  $SK$  and  $SQ$  will be in the same direction and so also will  $SK'$  and  $SQ'$ . Hence in this case  $TS$  will bisect the angle  $QSQ'$ .

Also, if  $Q$ ,  $Q'$  are on the same side of the directrix and  $T$  be on the opposite side; then  $SK$  and  $SQ$  will be in

opposite directions and so also will  $SK'$  and  $SQ'$ . Hence in this case also  $TS$  will bisect the angle  $QSQ'$ .

If, however, the points  $Q, Q'$  are on opposite sides of the directrix,  $SK$  and  $SQ$  will be in the same or opposite directions according as  $SK'$  and  $SQ'$  are in opposite directions or in the same direction. Hence in this case, that is when  $Q, Q'$  are on different branches of a hyperbola,  $TS$  will bisect the *exterior* angle  $QSQ'$ .

Thus, *if  $TQ, TQ'$  be tangents to a conic whose focus is  $S$ ,  $TS$  will bisect the angle  $QSQ'$  unless the conic is a hyperbola and  $Q, Q'$  are on opposite branches, in which case  $TS$  will bisect the exterior angle  $QSQ'$ .*

[The student should draw figures to illustrate the different cases.]

**Cor.** *If the tangents at the points  $Q, Q'$  of a conic intersect at  $T$ , and the chord  $QQ'$  cut a directrix in  $D$ ; then will  $DT$  subtend a right angle at the corresponding focus.*

From the above  $ST$  bisects the interior or the exterior angle  $QSQ'$  according as  $Q, Q'$  are on the same or on opposite sides of the directrix. Also, by Prop. VI.,  $SD$  bisects the exterior or the interior angle  $QSQ'$  according as  $Q$  and  $Q'$  are on the same or on different sides of the directrix.

Hence in all cases the lines  $ST$  and  $SD$  are at right angles.

**Ex. 1.** Having given a directrix of a conic and the tangent at a given point on the curve, shew that the locus of the focus corresponding to the given directrix is a circle.

**Ex. 2.** Having given a focus of a conic, two points on the conic and the tangent at one of those points; find the corresponding directrix.

**Ex. 3.** Having given a directrix of a conic, two points on the curve and the tangent at one of those points; find the corresponding focus.

**Ex. 4.** Construct a conic having given a focus, the eccentricity and the tangent at a given point.

**Ex. 5.** Find the focus of a conic having given the directrix, the eccentricity and the tangent at a given point.

**Ex. 6.**  $PN$  is the perpendicular from any point of a conic on its transverse axis, and  $NP$  produced cuts the tangent at an extremity of the latus-rectum in the point  $K$ . Shew that  $NK = SP$ .

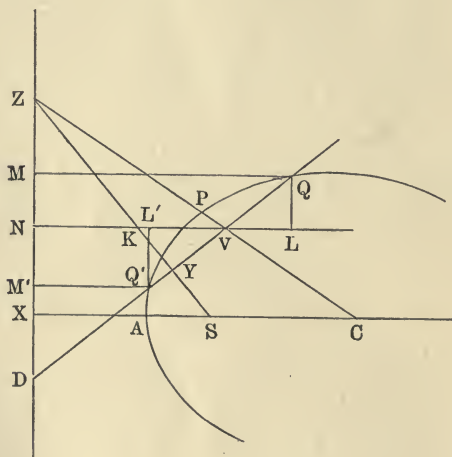
**18. Prop. XIII.** *The locus of the middle points of a system of parallel chords of a conic is a straight line through the centre of the conic.*

Let  $QQ'$  be one of the parallel chords, and let  $V$  be the middle point of the chord.

Draw  $QM, Q'M', VN$  perpendicular to the directrix, and  $QL, Q'L'$  perpendicular to  $NV$ .

Then, since  $V$  is the middle point of  $QQ'$ ,  $V$  is also the middle point of  $LL'$ .

Draw through the focus  $S$  a line perpendicular to  $QQ'$  meeting the corresponding directrix in  $Z$ ,  $QQ'$  in  $Y$ , and  $NV$  in  $K$ .



Then

$$\begin{aligned}
 SQ^2 - SQ'^2 &= YQ^2 - Q'Y^2 = (YQ + Q'Y)(YQ - Q'Y) \\
 &= 4YV \cdot VQ, \\
 [\text{since } YQ + Q'Y &= 2VQ \text{ and } YQ - Q'Y = 2YV] \\
 &= 4LV \cdot VK,
 \end{aligned}$$

since  $Q, L, Y, K$  are on a circle.

Again, if  $e$  be the eccentricity of the conic,

$$SQ^2 = e^2 \cdot QM^2 = e^2 \cdot LN^2,$$

and

$$SQ'^2 = e^2 \cdot Q'M'^2 = e^2 \cdot L'N^2.$$

$$\text{Hence } SQ^2 - SQ'^2 = e^2 (LN^2 - L'N^2) = 4e^2 \cdot LV \cdot VN.$$

$$\text{Hence } LV \cdot VK = e^2 \cdot LV \cdot VN;$$

$$\therefore VK = e^2 \cdot VN.$$

But [Art. 4]  $CS = e^2 \cdot CX$ , where  $C$  is the centre of the conic.

Hence  $ZVC$  is a straight line.

But  $Z$  is a fixed point for all chords parallel to  $QQ'$ , and therefore *the middle points of all parallel chords of a conic are on the fixed line joining the centre to the point where a directrix is met by a line through the corresponding focus perpendicular to the chords.*

**Def.** The locus of the middle points of a system of parallel chords of a conic is called a **diameter**.

From the above we see that all diameters of a conic are straight lines through the centre of the conic, and therefore in the particular case of a parabola the diameters are straight lines which cut the axis at an infinite distance from the focus, and are thus parallel to the axis. This result can also be deduced by putting  $e=1$  in the relation  $VK=e^2 \cdot VN$ , for if  $VK=VN$ ,  $Z$  and  $N$  must coincide.

**Cor. I.** *If any diameter of a conic meet the curve in  $P$ , then will the tangent at  $P$  be parallel to the system of chords bisected by that diameter.*

For if the line through  $P$  parallel to the chords bisected by the diameter  $CP$  meet the conic again in the point  $R$ , the middle point of  $PR$  will be on  $CP$ , and therefore at  $P$ , which can only be the case when the points  $P$  and  $R$  coincide.

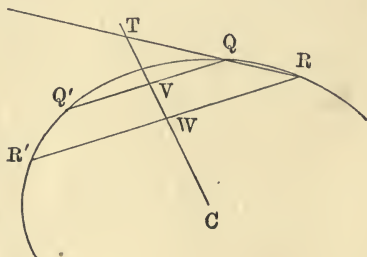
We know that the diameters  $ACA'$ ,  $BCB'$  bisect chords parallel to  $BB'$ ,  $AA'$  respectively; it therefore follows that *the tangents at  $A$  and  $A'$  are parallel to  $BCB'$ , and the tangents at  $B$  and  $B'$  are parallel to  $ACA'$ .*

Since a diameter meets a central conic in *two* points it follows that the tangents at those two points are parallel to one another.

This however follows immediately from the fact that the centre is the middle point of all chords of the conic which pass through it. For, if  $PCP'$ ,  $RCR'$  be any two chords,  $PR$  will be equal and parallel to  $P'R'$ . Hence if  $R$  move up to  $P$ , and therefore  $R'$  to  $P'$ , in the limit the tangent at  $P$  will be parallel to the tangent at  $P'$ .

**Cor. II.** *The tangents at the extremities of any chord of a conic meet on the diameter which bisects the chord.*

Let  $QQ'$ ,  $RR'$  be any two parallel chords of a conic; and let  $V$ ,  $W$  be their middle points. Then  $WV$  is a diameter of the curve. Let  $RQ$  cut  $WV$  in  $T$ .



Then  $WT : VT = RW : QV$ ;

$$\therefore WT : VT = R'W : Q'V.$$

Hence, as  $RW$  is parallel to  $Q'V$ ,  $R'Q'T$  is a straight line.

Thus  $RQ$ ,  $R'Q'$  meet on the diameter which bisects  $QQ'$ , and this is true for all positions of the parallel chord  $RR'$ .

Now let  $RR'$  move up to and ultimately coincide with  $QQ'$ ; then the lines  $RQ$ ,  $R'Q'$  will ultimately be the tangents at  $Q$ ,  $Q'$  respectively.

Hence the tangents at  $Q$ ,  $Q'$  will meet on the diameter which bisects  $QQ'$ .

Ex. 1. Having given an arc of a conic drawn on paper; find its centre by means of a ruler and compasses.

Ex. 2. Shew that the transverse axis is the greatest and the conjugate axis is the least diameter of an ellipse.

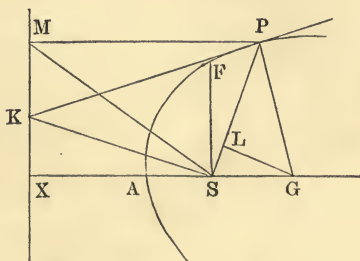
Ex. 3. Shew that the transverse axis is the longest and the latus-rectum the shortest focal chord of an ellipse.

**19. Prop. XIV.** *If the normal at any point  $P$  of a conic cut the transverse axis in  $G$ , then will  $SG : SP$  be equal to the eccentricity of the conic.*

Let the tangent at  $P$  cut the directrix corresponding to  $S$  in the point  $K$ .

Draw  $PM$  perpendicular to the directrix, and join  $SM$ .

Then, since the angles  $KSP$  and  $KMP$  are right angles,  $K, S, P, M$  are on a circle.



Hence  $\angle SMP = \angle SKP$

$= \text{compt. of } \angle SPK = \angle SPG.$

Also  $\angle SPM = \angle PSG$ , since  $PM$  and  $SG$  are parallel.

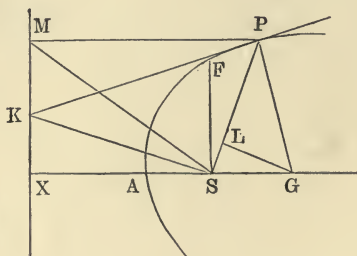
The triangles  $SPM, GSP$  are therefore similar, and we have

$$SG : SP = SP : PM = SA : AX.$$

**20. Prop. XV.** *If the normal at any point  $P$  of a conic cut the transverse axis in  $G$ ; then will the projection of  $PG$  on  $SP$  be equal to the semi-latus-rectum.*

Draw  $GL$  perpendicular to  $SP$ , and  $SF$  perpendicular to  $SX$  meeting the curve in  $F$ ; then  $SF$  is the semi-latus-rectum, and we have to prove that  $PL = SF$ .

We can prove, as in Prop. XIV., that the triangles  $GSP$ ,  $SPM$  are similar.



$$\begin{aligned} \text{Hence} \quad PG : SM &= SP : PM \\ &= SA : AX. \end{aligned}$$

But, since  $\angle GPL = \angle SMP = \angle XSM$ ,  
and  $\angle GLP = \text{rt. } \angle = \angle SXM$ ,  
the triangles  $GLP$  and  $SXM$  are similar.

$$\text{Hence} \quad PL : SX = PG : SM.$$

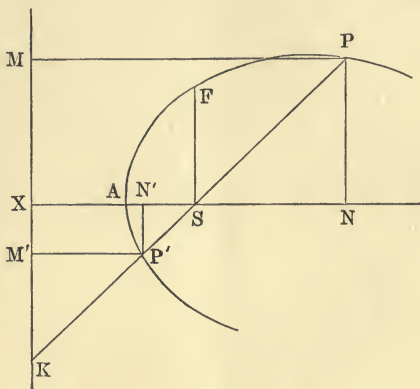
$$\begin{aligned} \text{Therefore} \quad PL : SX &= SA : AX \\ &= SF : SX. \end{aligned}$$

$$\text{Hence} \quad PL = SF = \text{semi-latus-rectum.}$$

**21. Prop. XVI.** *The semi-latus-rectum is the harmonic mean between the two segments of any focal chord.*

Let  $PSP'$  be any focal chord of a conic, and let  $PSP'$  produced, if necessary, cut the corresponding directrix in  $K$ . Let  $SF$  be the semi-latus-rectum.

Draw  $PM$ ,  $P'M'$  perpendicular to the directrix, and  $PN$ ,  $P'N'$  perpendicular to the transverse axis.



Then, since the triangles  $PSN$ ,  $P'SN'$  are similar,

$$\begin{aligned} SP : SP' &= SN : SN' \\ &= PM - SX : SX - P'M'. \end{aligned}$$

But  $SP = e \cdot PM$ ,  $SF = e \cdot SX$  and  $SP' = e \cdot P'M'$ ;

$$\therefore SP - SF = e(PM - SX),$$

and

$$SF - SP' = e(SX - P'M');$$

$$\therefore PM - SX : SX - P'M' = SP - SF : SF - SP';$$

$$\therefore SP : SP' = SP - SF : SF - SP'.$$

Thus  $SP$ ,  $SF$  and  $SP'$  are in Harmonic Progression.

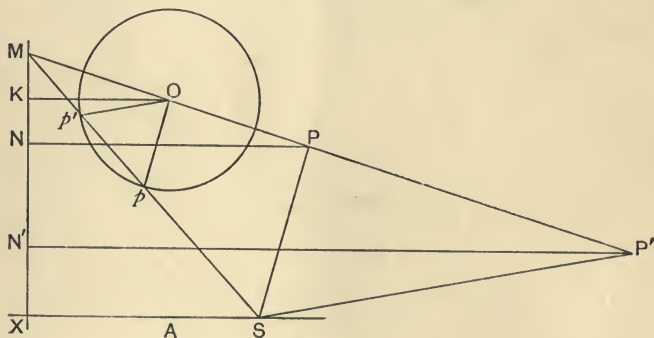


## THE ECCENTRIC CIRCLE.

**22. Definition.** The circle described with any point in the plane of a conic as centre and the ratio of whose radius to the perpendicular distance of the point from the directrix is equal to the eccentricity of the conic, is called the **eccentric circle** of the point.

**23.** By means of the eccentric circle we can find the points in which any given straight line cuts a conic whose focus, directrix and eccentricity are given.

For, let the line cut the directrix in  $M$ . Draw the eccentric circle of any point  $O$  on the line, and let  $SM$  cut the circle in  $p, p'$ . Join  $Op, Op'$ , and draw  $SP, SP'$  parallel to  $Op, Op'$  respectively, meeting the line  $MO$  in the points  $P, P'$  respectively. Then  $P, P'$  will be points on the conic.



Draw  $PN, P'N'$  and  $OK$  perpendicular to the directrix.

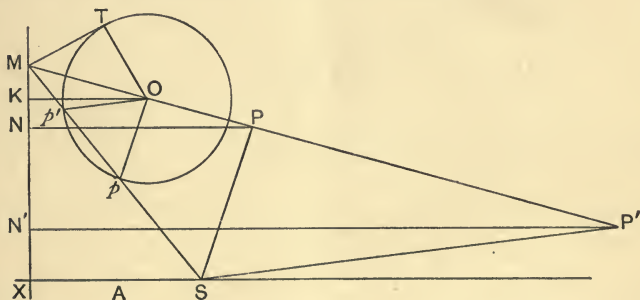
Then, since  $SP$  is parallel to  $Op$  and  $PN$  to  $OK$ ,

$$\begin{aligned} SP : Op &= PM : OM \\ &= PN : OK ; \\ \therefore SP : PN &= Op : OK. \end{aligned}$$

But, by the definition of the eccentric circle,  $Op : OK$  is equal to the eccentricity of the conic. Hence  $P$ , and similarly  $P'$ , is a point on the conic.

**24. Prop. XVII.** *The ratio of the rectangles contained by the segments of two intersecting chords of a conic, which are parallel to two fixed straight lines respectively, is independent of the position of the point of intersection of the chords.*

Let one of the chords through the point  $O$  cut the conic in  $P, P'$  and the directrix in  $M$ . Let the line  $SM$ , joining the focus to the point  $M$ , cut the eccentric circle of  $O$  in the points  $p, p'$ . Then, as in Art. 23,  $Op, Op'$  will be parallel to  $SP, SP'$  respectively.



Hence  $PO : OM = Sp : pM$ ,  
 and  $P'O : OM = Sp' : p'M$ ;  
 $\therefore PO \cdot P'O : OM^2 = Sp \cdot Sp' : Mp \cdot Mp'$ .

Hence  $PO \cdot P'O : Sp \cdot Sp' = OM^2 : MT^2$ , where  $MT$  is the tangent from  $M$  to the eccentric circle of  $O$ .

Now, if the line  $OPP'$  be fixed in direction, the ratio  $OM : OK$  will be constant. Hence, as  $OK : OT$  is equal to the eccentricity of the conic,  $OM : OT$  will be constant, and therefore, as  $OTM$  is a right angle,  $OM : MT$  will be constant provided that the chord is parallel to a fixed straight line.

Let another chord through  $O$  cut the conic in  $Q, Q'$  and the directrix in  $M'$ . Let  $M'T'$  be the tangent from  $M'$  to the eccentric circle of  $O$  and let  $SM'$  cut that circle in  $q, q'$ . Then, as before,

$$QO \cdot Q'O : Sq \cdot Sq' = OM'^2 : M'T'^2.$$

But, as before, the ratio  $OM'^2 : M'T'^2$  will be constant if the chord  $OQQ'$  be parallel to a second fixed straight line.

Also  $Sp \cdot Sp' = Sq \cdot Sq'$ ,

whence it follows that the ratio  $PO \cdot P'O : QO \cdot Q'O$  will be constant for all positions of  $O$ , provided that each of the chords is parallel to a fixed straight line.

### EXAMPLES.

1. Given the focus and directrix of a conic, shew that the chord of contact of the tangents drawn to the conic from a given point will pass through a fixed point.

2.  $PSP'$  is any focal chord of a conic, and  $Q$  is any other point on the curve;  $PQ, P'Q$  cut the directrix corresponding to the focus  $S$  in  $K, K'$  respectively. Shew that the rectangle  $KX \cdot XK'$  is constant.

3. The tangent at any point  $P$  of a conic whose focus is  $S$  meets the transverse axis in  $T$ . Shew that  $ST$  is greater than, equal to, or less than  $SP$  according as the conic is an ellipse, parabola, or hyperbola.

4.  $PSQ$  is a focal chord of a conic and the tangent at  $P$  is cut in  $T$  by the line through  $Q$  perpendicular to  $PQ$ . Shew that  $TQ$  is bisected by the directrix.

5. The focal chord  $PSP'$  of a conic, produced if necessary, cuts the directrix in  $K$ . Shew that  $SK$  is a harmonic mean between  $PS$  and  $P'K$ .

6. A chord  $QQ'$  of a conic cuts the directrix in  $Z$  and the line joining the points of contact of the tangents from  $Z$  to the conic in  $K$ . Shew that  $ZQ, ZK, ZQ'$  are in harmonic progression.

7. Having given the foci of a central conic and the line joining the points of contact of the tangents drawn to the conic from a given point; find the directrices.

8. Draw, parallel to a given straight line, a tangent to a conic whose focus, directrix and eccentricity are given.

9. The normal at the point  $P$  on a conic whose focus is  $S$  meets the transverse axis in  $G$ , and  $PM$  is the perpendicular on the directrix corresponding to the focus  $S$ . Shew that, for all positions of  $P$  on the curve,  $MS$  and  $PG$  intersect on a fixed straight line.

10. Shew that the tangent at a point of a conic is cut by the latus-rectum produced and the directrix in points whose distances from the corresponding focus are in the ratio of the eccentricity.

11. Shew that the tangents at the ends of any focal chord of a conic cut the latus-rectum produced in points which are equidistant from the focus.

12.  $QQ'$  is any chord of a conic which subtends a given angle at the focus; shew that the locus of the point of intersection of the tangents at  $Q$  and  $Q'$  is a second conic; and that the line  $QQ'$  touches a third conic, and also that the three conics have a common focus and directrix.

13. Two sides of a triangle are given in position, and the third side subtends a constant angle at a fixed point. Shew that the third side always touches a conic of which the fixed point is a focus.

14. On any tangent to a conic two points  $K, K'$  are taken such that  $KK'$  subtends a right angle at a focus; shew that the other tangents from  $K, K'$  intersect on the corresponding directrix.

15. The tangent at a point  $P$  of a conic whose focus is  $S$  cuts the directrix in  $K$  and the transverse axis in  $T$ , and  $PM$  is the perpendicular from  $P$  on the directrix. Shew that  $SM$  touches the circle  $SKT$ .

16. Shew that, if  $SY$  be the perpendicular on the tangent at any point  $P$  of a conic whose focus is  $S$ , and if  $PM$  be the perpendicular on the corresponding directrix, the triangles  $SYX$  and  $SPM$  will be similar.

17. The tangents at the extremities of a focal chord  $PSP'$  of a conic meet in  $T$ , and  $P, P'$  are on opposite sides of  $S$ ; shew  $ST^2 \begin{matrix} > \\ < \end{matrix} PS \cdot SP'$ , according as the conic is an ellipse, parabola or a hyperbola.

18.  $PSP'$  is a focal chord of a conic, and the normals at  $P, P'$  meet in  $O$ ; shew that the line through  $O$  parallel to the transverse axis will bisect  $PP'$ .

19.  $PSP'$  is a focal chord of a conic, and the normals at  $P, P'$  meet in  $O$ , and from  $O$  the line  $OK$  is drawn perpendicular to  $PSP'$ . Shew that  $SP = P'K$  and  $SP' = PK$ .

20. The normals at any two points  $P, P'$  of a conic cut the transverse axis in  $G, G'$  respectively. Shew that the projections of  $PG$  and  $P'G'$  on  $PP'$  are equal to one another.

## CHAPTER II.

### THE PARABOLA.

**25. Definitions.** A **parabola** is the locus of a point which moves in the plane containing a given point and a given straight line, in such a manner that its distance from the given point is equal to its perpendicular distance from the given straight line. The given point is called the **focus** and the given straight line is called the **directrix** of the parabola.

**Prop. I.** *Having given the focus and directrix of a parabola, to find any number of points on the curve.*

Let  $S$  be the focus and  $MM'$  the directrix of a parabola.

Draw through  $S$  the line  $XSN$  perpendicular to the directrix, meeting the directrix in  $X$ . Bisect  $XS$  in  $A$ ; then, since  $SA = AX$ ,  $A$  is a point on the curve.

Through any point  $N$  on  $XSN$  draw  $LNL'$  perpendicular to  $XSN$ .

With centre  $S$  and radius equal to  $XN$  describe a circle to cut  $LNL'$  in the points  $P, P'$ .

Draw  $PM, P'M'$  perpendicular to the directrix.

Then, since  $PM, NX, P'M'$  are all perpendicular to  $MXM'$  and  $PNP'$ , we have

$$SP = XN = MP, \text{ and } SP' = XN = M'P'.$$

Hence  $P$  and  $P'$  are points on the curve.

The necessary and sufficient condition that the circle centre  $S$  and radius  $XN$  should cut the line  $NL$  is that

$XN$  should be greater than  $SN$ , and this will be the case provided  $N$  be taken anywhere to the right of  $A$  in the

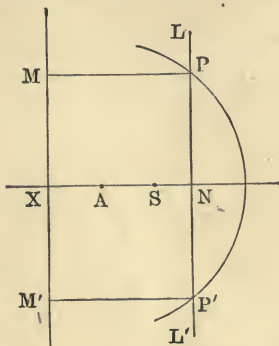


figure. Thus any straight line parallel to the directrix of the parabola, and on the same side of the directrix as the focus, will cut the curve in two points provided the distance of the line from the directrix be not less than half the distance of the focus from the directrix.

Hence *a parabola lies entirely on the same side of the directrix as the focus, and extends to an unlimited distance.*

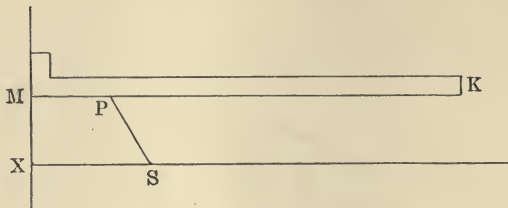
Since  $SP = SP'$ , and  $XSN$  is perpendicular to  $PP'$ ,  $PN$  must be equal to  $NP'$ . Now when a straight line is so related to a curve that corresponding to any point of the curve there is another point such that the chord joining the two points is bisected perpendicularly by the straight line, then the curve is said to be **symmetrical** about the straight line, and the straight line is called an **axis** of the curve.

We have thus proved that *a parabola is symmetrical about the straight line through its focus perpendicular to its directrix*, which line is accordingly called the **axis** of the curve.

A point where an axis cuts the curve is called a **vertex**.

Thus, in the figure, the point  $A$  is the vertex of the parabola.

**26.** We have shewn how to determine any number of points on the curve. The curve may be described continuously in the following manner.



Let a straight rod  $MK$  be made to slide with one end  $M$  on the directrix and so that the rod is always perpendicular to the directrix. Then, if a string whose length is equal to that of the rod have one end fastened at the extremity of the rod and the other at the focus, and if the string be kept constantly stretched by a pencil in contact with the rod, the point of the pencil will describe a parabola with the given focus and directrix. For, since  $SP + PK = KM$ ,  $SP = PM$ .

**27.** It is easily seen that  $SP$  is less than  $PM$  for any point  $P$  within a parabola, and that  $SP$  is greater than  $PM$  for all points outside.

For, if  $P$  be within the curve, and  $PM$  be drawn perpendicular to the directrix it will cut the curve at some point  $Q$  between  $P$  and  $M$ . Then  $SQ = QM$ , hence  $PM = SQ + QP$ , and  $SQ + QP$  is greater than  $SP$ .

Similarly for a point outside the curve.

**Ex. 1.** If the focal distances of two points on a parabola be equal to one another, the line joining the points must be parallel to the directrix, and the focal distances of the points must be equally inclined to the axis.

**Ex. 2.** If the directrix of a parabola be given and also one point on the curve, the focus must lie on a fixed circle.

**Ex. 3.** Find the focus of a parabola when the directrix and two points on the curve are given. How many parabolas can be drawn to satisfy the given conditions?

**Ex. 4.** If the focus of a parabola be given and one point on the curve, the directrix will touch a fixed circle.

**Ex. 5.** Find the directrix of a parabola when the focus and two points on the curve are given. How many solutions will there be?

**Ex. 6.** Find the directrix of a parabola having given the focus, the direction of the axis, and one point on the curve.

Ex. 7. The locus of the centre of a circle which passes through a given point and touches a given straight line, is a parabola.

Ex. 8. The locus of the centre of a circle which touches a given straight line and a given circle is a parabola whose focus is at the centre of the given circle, and whose directrix is parallel to the given straight line and at a distance from it equal to the radius of the given circle.

**28. Definitions.** The perpendicular from any point of a parabola on the axis is called the **ordinate** of the point. The length of the axis, measured from the vertex, cut off by the ordinate of any point, is called the **abscissa**.

Thus, in the figure Art. 25,  $PN$  is the *ordinate* of the point  $P$ , and  $AN$  is the *abscissa*.

A chord  $PP'$  perpendicular to the axis is sometimes called a *double ordinate*.

Any chord through the focus is called a *focal chord*.

The focal chord perpendicular to the axis is called the **latus-rectum**.

**29. Prop. II.** *The length of the latus-rectum of a parabola is four times the distance of the focus from the vertex.*

$$[LL' = 4AS.]$$

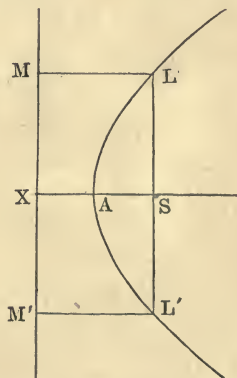
Let  $S$  be the focus,  $MXM'$  the directrix,  $XAS$  the axis and  $LSL'$  the latus-rectum. Draw  $LM$ ,  $L'M'$  perpendicular to the directrix.

Then, by definition,

$$SL = LM = SX,$$

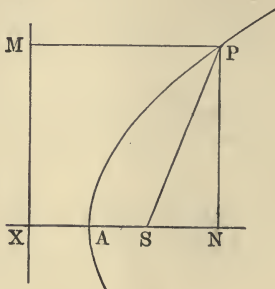
and  $SL' = L'M' = SX.$

Hence  $LL' = 2SX = 4SA.$



It should be noticed that two parabolas which have equal latera-recta are equal in all respects. For, since the latera-recta are equal, the distances of the foci from the respective directrices must likewise be equal. One curve may therefore be superimposed on the other (as in Euclid I. 4) so that the directrices are coincident and also the foci; and in that case the two parabolas will altogether coincide.

**30. Prop. III.** *The ordinate at any point of a parabola is a mean proportional to the abscissa and the latus-rectum. [ $PN^2 = 4AS \cdot AN$ .]*



Join  $SP$ , and draw  $PM$  perpendicular to the directrix, and  $PN$  perpendicular to the axis.

Then, since  $SP^2 = PM^2 = XN^2$ ;

and  $SP^2 = PN^2 + SN^2$ ;

$$\begin{aligned}\therefore PN^2 &= XN^2 - SN^2 \\ &= (XN - SN)(XN + SN) \\ &= 4AS \cdot AN,\end{aligned}$$

since  $XN + SN = 2AN$  and  $XN - SN = 2AS$ .

From the relation  $PN^2 = 4AS \cdot AN$  it follows that the square of the perpendicular from any point of a parabola on the axis varies as the perpendicular on a line through the vertex parallel to the directrix.

**Conversely.** *If a point move in a plane in such a manner that the square of the perpendicular from it on one of two fixed straight lines at right angles to each other varies as the perpendicular on the other, the point will describe a parabola.*

**Ex.** If  $PN$  is the ordinate of any point  $P$  of a parabola, the locus of the middle point of  $PN$  is a parabola.

For, if  $Q$  be the middle point of  $NP$ ,  $NQ^2 = \frac{1}{4}NP^2 = AS \cdot AN$ . Hence the locus of  $Q$  is a parabola whose vertex is  $A$ , whose axis is  $ASN$ , and whose latus-rectum is equal to  $AS$ .

Ex. 1. When the ordinate and the abscissa of a point on a parabola are equal to one another, each is equal to the latus-rectum.

Ex. 2. Find the abscissa corresponding to a double ordinate whose length is three times the latus-rectum.

Ex. 3.  $PNP'$  is a double ordinate of a parabola whose vertex is  $A$ , and the circle  $APP'$  cuts the axis in  $Q$ . Shew that  $NQ$  is equal to the latus-rectum.

Ex. 4.  $PM$  is the perpendicular from any point  $P$  of a parabola upon its directrix, and on  $PM$  is taken a point  $Q$  such that  $PQ$  is constant. Shew that the locus of  $Q$  is an equal parabola.

Ex. 5. The locus of a point which divides the ordinate of a parabola in a constant ratio is another parabola with the same vertex and axis.

Ex. 6.  $A$  is the vertex of a parabola and  $P$  is any point on the curve. Shew that the locus of the middle point of  $AP$  is another parabola.

Ex. 7. If  $S$  be the focus of a parabola and  $P$  be any point on the curve, the locus of the middle point of  $SP$  is a parabola whose focus is  $S$  and whose directrix is midway between  $S$  and the directrix of the given parabola.

Ex. 8.  $S$  is the focus of a parabola and  $P$  is any point on the curve;  $Q$  is the point on  $SP$  such that  $SQ : SP$  is equal to a given ratio. Shew that the locus of  $Q$  is a parabola.

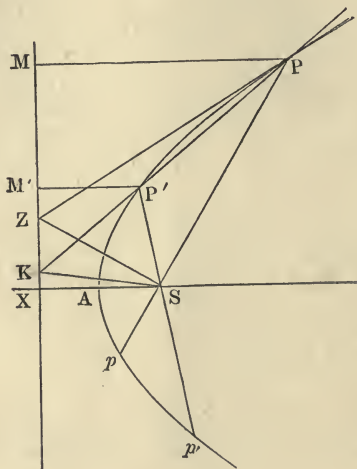
Ex. 9.  $PM$  is the perpendicular from any point  $P$  of a parabola on its directrix, and  $MP$  is produced to  $Q$  so that  $MP = PQ$ . Shew that, if  $N, N'$  be the feet of the ordinates through  $P, Q$  respectively,  $SN' = 2AN$ , and that the locus of  $Q$  is a parabola whose vertex is  $S$ .

Ex. 10.  $PM$  is the perpendicular upon the directrix of a parabola from any point  $P$  on the curve. Shew that the locus of  $Q$ , the middle point of  $MP$ , is a parabola whose vertex is midway between  $X$  and  $A$ .

**31. Definitions.** Let two points  $P, P'$  be taken on any curve, and let the point  $P'$  move along the curve nearer and nearer to the point  $P$ ; then the limiting position of the line  $PP'$ , when  $P'$  moves up to and ultimately coincides with  $P$ , is called the **tangent** to the curve at the point  $P$ . Also the line through any point  $P$  of a curve perpendicular to the tangent thereat is called the **normal** to the curve at the point  $P$ .

**32. Prop. IV.** (i) *If a straight line cut the directrix of a parabola, whose focus is  $S$ , in  $K$  and the curve in  $P, P'$ ; then will  $KS$  bisect the exterior angle between  $SP$  and  $SP'$ .*

(ii) *The portion of any tangent to a parabola, intercepted between its point of contact and the directrix, subtends a right angle at the focus.*



Join  $SP, SP'$  and produce  $PS, P'S$  to  $p, p'$ . Draw  $PM, P'M'$  perpendicular to the directrix.

Then, from the similar triangles  $KPM, KP'M'$ , we have

$$\begin{aligned} KP : KP' &= PM : P'M' \\ &= SP : SP', \text{ by def.} \end{aligned}$$

Hence  $KS$  bisects the angle  $PSp'$  or  $P'Sp$ .....(i).

Now let  $P'$  move up to and ultimately coincide with  $P$ , and let  $PZ$  be the ultimate position of the line  $PP'$ , that is of the *tangent* at  $P$ . Then  $KS$  will always make equal angles with  $P'S$  and  $Sp$ , and therefore ultimately

$ZS$  will make equal angles with  $PS$  and  $Sp$ . Hence each of the angles  $ZSP$ ,  $ZSp$  is a right angle.

Thus  $ZP$  subtends a right angle at  $S$ .....(ii).

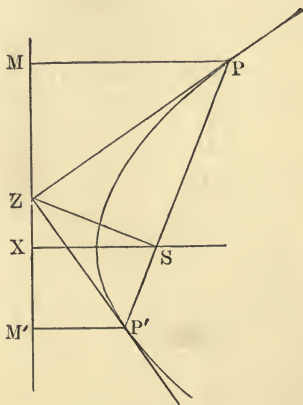
**Conversely**, if  $SZ$  be drawn perpendicular to  $SP$  to meet the directrix in  $Z$ , then  $PZ$  will be the tangent at  $P$ .

**33. Prop. V.** *Tangents at the extremities of a focal chord of a parabola will intersect at right angles in the directrix.*

Let  $PSP'$  be any focal chord. Draw  $SZ$  perpendicular to  $PSP'$  meeting the directrix in  $Z$ .

Then, since the angles  $ZSP$ ,  $ZSP'$  are right angles,  $ZP$  and  $ZP'$  are tangents.

[Converse of Prop. IV.]



Thus the tangents at the extremities of the focal chord intersect in the directrix.

Draw  $PM$ ,  $P'M'$  perpendicular to the directrix.

Then, since  $ZSP$  and  $ZMP$  are right angles, the points  $Z$ ,  $S$ ,  $P$ ,  $M$  are on a circle; and, since  $SP$ ,  $PM$  are equal chords of this circle, the subtended angles  $SZP$  and  $MZP$  are equal. Therefore  $ZP$  bisects the angle  $MZS$ .

Similarly  $ZP'$  bisects the angle  $M'ZS$ .

Hence  $ZP$  and  $ZP'$  are at right angles.

Ex. 1.  $PM$  is the perpendicular on the directrix of a parabola, whose focus is  $S$  and vertex  $A$ ; shew that  $MS$  bisects the angle  $ASP$ .

Ex. 2. If  $PA$  cut the directrix in  $K$ ,  $KS$  bisects the exterior angle  $ASP$ .

Ex. 3.  $P$  is any point on a parabola whose vertex is  $A$ ,  $PM$  is the perpendicular on the directrix, and  $PA$  cuts the directrix in  $K$ . Shew that  $MK$  subtends a right angle at the focus.

Ex. 4.  $PSP'$  is any focal chord of a parabola, and  $Q$  any point on the curve.  $PQ$ ,  $P'Q$  cut the directrix in  $K$ ,  $K'$  respectively. Shew that  $KK'$  subtends a right angle at the focus.

Ex. 5.  $PSP'$  is a focal chord of a parabola whose vertex is  $A$ , and  $PA$  meets the directrix in  $M$ ; shew that  $P'M$  is parallel to the axis.

Ex. 6. If two parabolas have a common directrix the line joining their common points will bisect at right angles the line joining their foci.

Ex. 7. If three parabolas have a common directrix the three common chords of the parabolas taken in pairs will meet in the centre of the circum-circle of the triangle formed by their foci.

Ex. 8. The tangents at the extremities of the latus-rectum of a parabola pass through the foot of the directrix.

Ex. 9. A series of parabolas have a common directrix and axis; shew that they all touch two fixed straight lines at right angles to one another.

Ex. 10. Any number of parabolas have a common directrix and touch a given straight line; shew that they all touch another straight line perpendicular to the former, and that their foci lie on a fixed straight line.

Ex. 11.  $MKM'$  is the directrix of a parabola whose focus is  $S$ . Shew that the lines bisecting the angles  $MKS$ ,  $M'KS$  will touch the parabola,  $K$  being any point on the directrix.

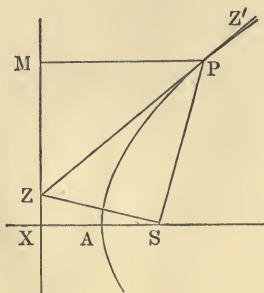
Ex. 12.  $S$ ,  $S'$  are the foci of two parabolas which have a common directrix; shew that their common tangents meet at right angles at the point of intersection of  $SS'$  and the common directrix.

Ex. 13. Shew that the circle described on any focal chord of a parabola as diameter will touch the directrix.

Ex. 14. Having given the directrix of a parabola and the tangent at a given point; find the focus.

Ex. 15. Having given the directrix and two tangents to a parabola; find the focus. When are the conditions insufficient?

**34. Prop. VI.** *The tangent at any point P of a parabola bisects the angle between SP and the perpendicular PM on the directrix.*



Let the tangent cut the directrix in Z. Join SZ, and draw PM perpendicular to the directrix.

Then, since the angles ZSP, ZMP are right angles, the points Z, S, P, M are on a circle; and, since SP, PM are equal chords of this circle,

$$\angle SZP = \angle MZP.$$

Hence the complements of these angles are equal, that is

$$\angle SPZ = \angle MPZ.$$

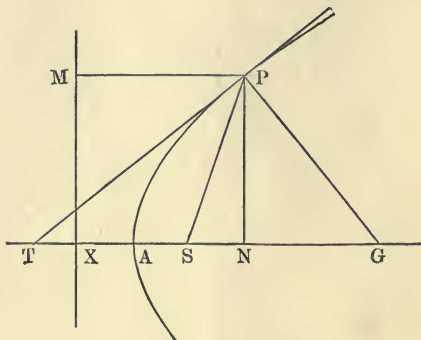
Hence PZ bisects the angle SPM.

**Cor. 1.** The tangent at A bisects the angle between SA and AX. Hence the tangent at the vertex is perpendicular to the axis.

**Cor. 2.** If ZP be produced to Z', the angles SPZ' and MPZ' will be equal.

**35. Prop. VII.** *If the tangent at any point  $P$  of a parabola meet the axis in  $T$ , and  $PN$  be the ordinate of  $P$ ; then will*

$$SP = ST \text{ and } TA = AN.$$



Join  $SP$ , and draw  $PM$  perpendicular to the directrix.

Then  $\angle SPT = \angle MPT$ ; [Prop. VI.

$$\therefore \angle SPT = \angle PTS;$$

[since  $PM$  and  $NT$  are parallel.

$$\therefore TS = SP.$$

And, since  $TS = SP = PM = XN$ ;

$$\therefore TA + AS = XA + AN;$$

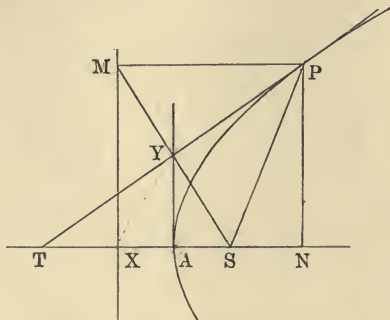
$$\therefore TA = AN.$$

**Def.** The portion of the axis intercepted by the ordinate of any point and the corresponding tangent is called the **sub-tangent**.

Hence, in a parabola, the subtangent is equal to twice the abscissa.



**37. Prop. IX.** *The locus of the foot of the perpendicular from the focus on a tangent to a parabola is the tangent at the vertex, and the length of the perpendicular is a mean proportional between the focal distances of the point of contact and the vertex.*



Join  $SP$ . Draw  $PM$  perpendicular to the directrix, and join  $SM$  cutting the tangent at  $P$  in  $Y$ .

Then, since  $SP = PM$ , and the tangent at  $P$  bisects the angle  $SPM$ , the tangent must be perpendicular to  $SM$  and will bisect  $SM$  in  $Y$ .

Then, since  $SY = YM$  and  $SA = AX$ ,  $AY$  must be parallel to  $XM$ .

Thus  $Y$  is on the tangent at the vertex.

Again, since  $SYT$  and  $YAS$  are right angles, the triangles  $ASY$ ,  $YST$  are similar;

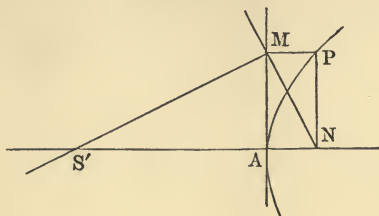
$$\therefore AS : SY = SY : ST = SY : SP.$$

Hence

$$SY^2 = AS \cdot SP.$$

The converse of this proposition is very important, namely that if a straight line move in such a manner that the foot of the perpendicular upon it from some fixed point always lies on a fixed straight line; then the moving line will always touch the parabola whose focus is the fixed point and of which the fixed straight line is the tangent at the vertex.

Ex.  $PN$ ,  $PM$  are the perpendiculars from any point  $P$  of a parabola on the axis and the tangent at the vertex of the parabola; shew that  $NM$  touches a fixed parabola.



Draw  $MS'$  perpendicular to  $NM$  meeting the axis in  $S'$ .  
Then, since  $NMS'$  and  $MAN$  are right angles,

$$MA^2 = S'A \cdot AN.$$

But

$$MA^2 = PN^2 = 4AS \cdot AN.$$

Hence  $S'A = 4AS$ , so that  $S'$  is a fixed point.

Hence  $MN$  touches the parabola whose focus is  $S'$  and of which  $AM$  is the tangent at the vertex.

# EXAMPLES. VII. AND VIII.

1. Shew that  $TSPM$  is a rhombus.
2. Shew that  $TP$  and  $SM$  bisect each other at right angles.
3. Shew that  $MPGS$  is a parallelogram.
4. If  $SPG$  is an equilateral triangle each side is equal to the latus-rectum.
5. Shew that the triangles  $TMS$  and  $SPG$  are equal in all respects.
6. Shew that the triangles  $TXM$  and  $SNP$  are equal in all respects.
7. Shew that the triangles  $MXS$  and  $PNG$  are equal in all respects.
8. If  $GL$  be perpendicular to  $SP$ ; then will  $PL = NG = 2AS$ .
9. Shew that the tangents and normals at the ends of the latus-rectum are along the sides of a square.
10. If  $XH$  parallel to  $SP$  meet  $PM$  in  $H$ ; then  $NH$  will be parallel to  $PG$ .
11. Shew that the locus of the middle point of  $PG$  is a parabola whose vertex is  $S$ .

12. Any number of parabolas have the same focus and axis, and tangents are drawn to them from the same point on the common axis; shew that the points of contact of the tangents lie on a circle.

13. The line joining any point  $P$  of a parabola to the vertex cuts the directrix in  $K$ . Shew that  $KS$  is parallel to the tangent at  $P$ .

### EXAMPLES. IX.

1. Any tangent to a parabola meets the directrix and the latus-rectum produced in points equidistant from the focus.

2. If the focus  $S$  of a parabola be joined to any point  $M$  of the directrix, the lines which bisect  $SM$  at right angles will touch the curve.

3. Tangents are drawn to a series of concentric circles at the points where they are cut by a fixed straight line; shew that these tangents all touch a parabola.

4. Having given two tangents to a parabola and the focus, find the directrix.

5. Having given the focus of a parabola and two tangents, find the points of contact of the tangents.

6. Having given the focus of a parabola and the tangent at a given point, find the directrix.

7. If two parabolas have a common focus, their common chord will bisect the angle between their directrices.

8. If two parabolas have a common focus, the line joining it to the point of intersection of the directrices will be perpendicular to their common tangent.

9. If two equal parabolas have a common focus, their common chord will pass through the focus and will be perpendicular to their common tangent.

10.  $O$  is a fixed point,  $P$  is any point on a fixed straight line, and on the line a point  $Q$  is taken such that  $PQ=OP$ . Shew that the line through  $P$  and the middle point of  $OQ$  touches a parabola whose focus is  $O$ .

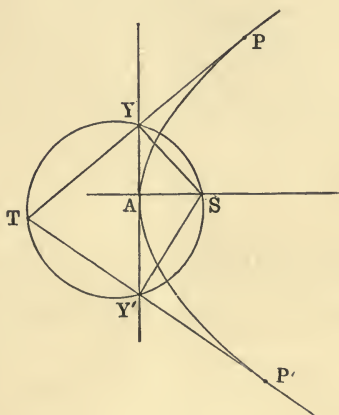
11.  $PN$  is the ordinate of any point  $P$  of a parabola, and a point  $M$  is taken on the axis such that  $AN=NM$ . Shew that  $MP$  touches a fixed parabola.

12. A line is drawn through the focus of a parabola so as to meet the tangent at a point  $P$  at a given angle; shew that the locus of the point of intersection, for different positions of  $P$ , is a straight line.

[The intersection is on the tangent which makes the given angle with the axis.]

**38. Prop. X.** *To draw tangents to a parabola from any external point.*

Let  $T$  be the external point. Join  $TS$ , and upon  $TS$  as diameter describe a circle cutting the tangent at the vertex in  $Y, Y'$ . Then the angles  $TY'S, TYS$  are right angles, and hence  $TY, TY'$  when produced will touch the parabola, for the feet of the perpendiculars from the focus on these lines are on the tangent at the vertex.



A method of drawing tangents which is applicable to all conics is given in Art. 16.

**Ex. 1.** If  $P$  be any point on a parabola whose focus is  $S$ , the circle whose diameter is  $SP$  will touch the tangent at the vertex of the parabola.

**Ex. 2.** If  $T$  be any point without a parabola, the circle whose diameter is  $ST$  will cut the tangent at the vertex of the parabola in two real points.

**Ex. 3.** Two real tangents can be drawn to a parabola from any external point.



**Cor. I.** The angle  $tTP'$  is supplementary to the sum of  $STP'$  and  $STP$ , that is to the sum of  $SPT'$  and  $STP$ .

Hence  $\angle tTP' = \angle TSP = \angle TSP'$ .

Thus *the exterior angle between two tangents to a parabola is equal to the angle either tangent subtends at the focus.*

**Cor. II.** Since the triangles  $PST$ ,  $TSP'$  are similar,

$$TP^2 : TP'^2 = PS^2 : ST^2.$$

Also  $PS : ST = ST : SP'$ ;

$$\therefore PS^2 : ST^2 = PS : SP'.$$

Hence  $TP^2 : TP'^2 = PS : SP'$ .

Thus, *the ratio of the squares on any two tangents to a parabola is equal to the ratio of the focal distances of their points of contact.*

### EXAMPLES.

1. If  $TP$ ,  $TP'$  be tangents to a parabola, and  $PP'$  meet the directrix in  $Z$ ; then will  $SZ$  be perpendicular to  $ST$ .

[For  $SZ$  bisects the exterior angle  $PSP'$ , and  $ST$  bisects the angle  $PSP'$ .]

2. If  $TP$ ,  $TP'$  be tangents to a parabola whose focus is  $S$ , the circles  $TPS$ ,  $TP'S$  will touch  $TP'$ ,  $TP$  respectively.

3. Having given two tangents to a parabola and the point of contact of one of them; shew that the locus of the focus is a circle.

4. Having given two points on a parabola and the tangents thereat, construct the curve [that is, find the focus and directrix].

5. From any point on the axis of a parabola two tangents are drawn; shew that these tangents cut any other tangent in points equidistant from the focus.

6. A variable tangent to a parabola cuts two fixed tangents in the points  $T$ ,  $T'$ ; shew that  $ST : ST'$  is constant,  $S$  being the focus.

7. Having given three tangents to a parabola, and the point of contact of one of them; construct the curve.

8. Having given three tangents to a parabola, and the direction of the axis; construct the curve.

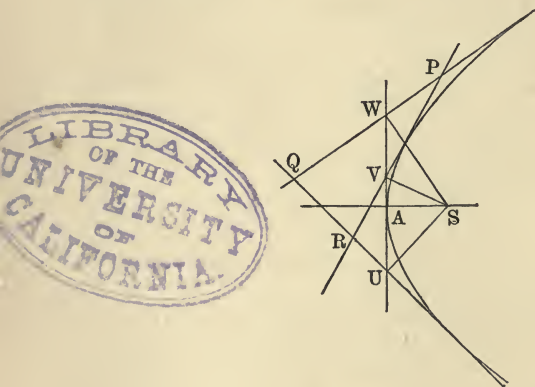
**40. Prop. XII.** *If the sides of a triangle touch a parabola, the circum-circle of the triangle will pass through the focus of the parabola.*

We know that the feet of the perpendiculars from the focus of a parabola lie on a straight line, namely on the tangent at the vertex of the parabola.

But it is a well-known geometrical theorem that if the feet of the perpendiculars from a point on the sides of a triangle lie on a straight line, that point must be on the circum-circle of the triangle.

The theorem may also be proved as follows.

Let  $PQR$  be the triangle formed by the tangents, and let the tangent at the vertex of the parabola meet  $QR$ ,  $RP$ ,  $PQ$  respectively in  $U$ ,  $V$ ,  $W$ .



Then, since  $SUQ$  and  $SWQ$  are right angles,  $S$ ,  $U$ ,  $W$ ,  $Q$  are cyclic, and therefore

$$\angle SQW = \angle SUV.$$

And, since  $SUR$  and  $SVR$  are right angles,  $S$ ,  $U$ ,  $V$ ,  $R$  are cyclic, and therefore

$$\angle SUV = \angle SRV.$$

Hence  $\angle SQW = \angle SRV$ , i.e.  $\angle SQP = \angle SRP$ ,  
whence it follows that  $S$ ,  $Q$ ,  $P$ ,  $R$  are cyclic.

**41. Prop. XIII.** *The middle points of a system of parallel chords of a parabola lie on a straight line parallel to the axis of the parabola.*

Let  $QQ'$  be any one of the system of parallel chords. Draw  $QM, Q'M'$  perpendicular to the directrix.

Then the circles whose centres are  $Q, Q'$  and radii  $QM, Q'M'$  respectively will touch the directrix in  $M, M'$  respectively and will pass through the focus.

But it is known that the common chord of any two circles is perpendicular to the line joining their centres and bisects any common tangent.

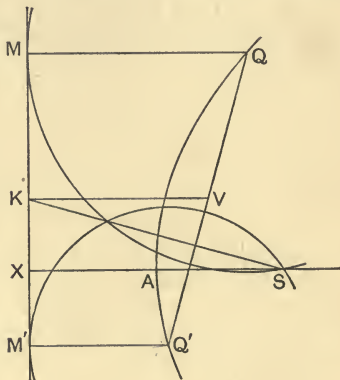
Hence the line through  $S$  perpendicular to  $QQ'$  will bisect  $MM'$ , in  $K$  suppose.

Then, since  $K$  is the middle point of  $MM'$ , the line through  $K$  parallel to the axis of the parabola, and therefore parallel to  $MQ$  and  $M'Q'$ , will bisect  $QQ'$ , in  $V$  suppose.

The point  $K$  is the same for all chords parallel to  $QQ'$ , whence it follows that *the middle points of a system of parallel chords of a parabola are on a straight line parallel to the axis.*

**Cor.** If the line through  $V$  parallel to the axis meet the curve in  $P$ , then the tangent at  $P$  will be parallel to the system of chords.

For, if the line through  $P$  parallel to the chords whose middle points are on  $PV$ , cut the curve again in  $R$ ; then the middle point of  $PR$  will be on  $PV$ , and therefore at the point  $P$ , which can only be the case when the points  $P, R$  coincide.



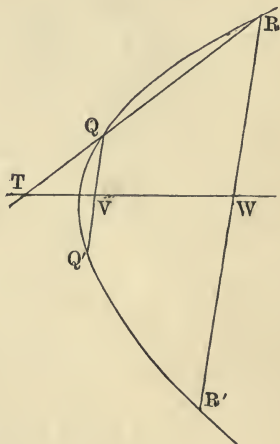
**Def.** The locus of the middle points of a system of parallel chords of a parabola is called a **diameter**.

From the above we see that *all diameters of a parabola are straight lines parallel to the axis of the parabola, and that the tangent at the point where a diameter meets the curve is parallel to the system of chords bisected by that diameter.*

**42. Prop. XIV.** *The tangents at the extremities of any chord of a parabola meet on the diameter which bisects the chord.*

Let  $QQ'$ ,  $RR'$  be any two parallel chords of a parabola; and let  $V$ ,  $W$  be their middle points. Then  $VW$  is a diameter of the curve.

Let  $QR$  cut  $VW$  in  $T$ .



Then

$$WT : VT = RW : QV;$$

$$\therefore WT : VT = WR' : VQ'.$$

Hence  $TQ'R'$  is a straight line.

Thus  $QR$ ,  $Q'R'$  meet on the diameter through  $V$ ; and this is true for all positions of the parallel chord  $RR'$ .

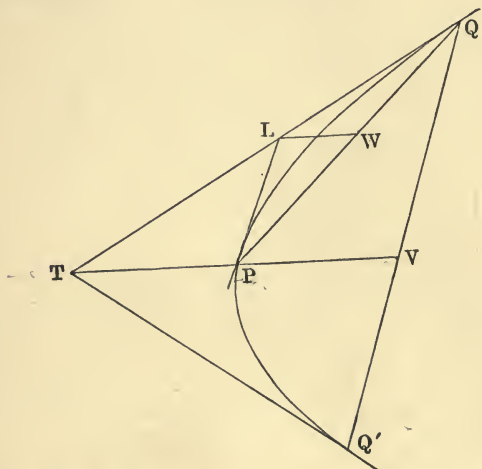
Now let  $RR'$  move up to and ultimately coincide with  $QQ'$ ; then the lines  $QR$ ,  $Q'R'$  will ultimately be the tangents at  $Q$ ,  $Q'$  respectively.

Hence the tangents at  $Q$ ,  $Q'$  meet on the diameter through  $V$ .

**Def.** A line  $QV$ , drawn parallel to the tangent at the extremity of any diameter  $PV$  of a parabola, is called **an ordinate to that diameter**.

The portion of any diameter, measured from its extremity, cut off by any ordinate is called its **abscissa**.

**43. Prop. XV.** *If the tangents at the extremities of any chord  $QQ'$  of a parabola meet in  $T$ , and if the diameter through  $T$  cut the curve in  $P$  and  $QQ'$  in  $V$ ; then will  $TP = PV$ .*



We know that the diameter through  $T$  will bisect  $QQ'$ , and that the tangent at  $P$  is parallel to  $QQ'$ .

Let the tangent at  $P$  cut  $TQ$  in  $L$ , and let the diameter through  $L$  cut  $PQ$  in  $W$ .

Then  $LW$  bisects  $PQ$  and is parallel to  $TPV$ .

Hence  $TL : LQ = PW : WQ$ ;

$$\therefore TL = LQ.$$

But  $PL$  is parallel to  $QV$ ;

$$\therefore TP : PV = TL : LQ;$$

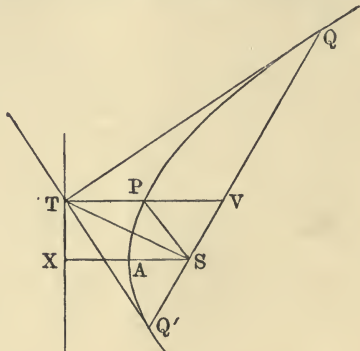
$$\therefore TP = PV.$$

**Cor.** Since  $TL = LQ$ , it follows that

$$LQ : LP = LT : LP.$$

Hence the ratio of the lengths of two tangents to a parabola is equal to the ratio of the sides of any triangle whose sides are parallel to the tangents and base parallel to the axis of the parabola.

**44. Prop. XVI.** The length of the focal chord which is parallel to the tangent at any point  $P$  of a parabola is  $4SP$ .



Let  $QSQ'$  be the focal chord parallel to the tangent at  $P$ . Then the tangents at  $Q, Q'$  intersect at right angles at a point which is on the directrix and also on the diameter through  $P$ .

Hence  $V$  is the middle point of the hypotenuse of the right-angled triangle  $QTQ'$ ;

$$\therefore QQ' = 2QV = 2VT.$$

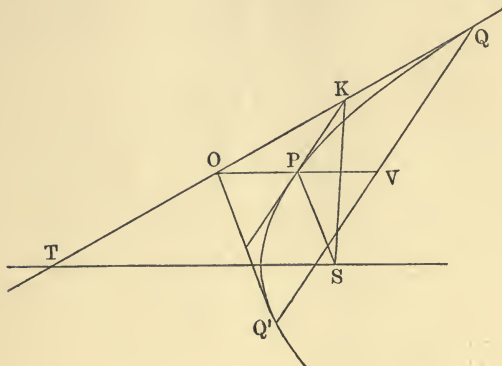
Also  $ST$  is perpendicular to  $QQ'$ , and  $P$  is the middle point of  $TV$ . Hence  $P$  is the middle point of the hypotenuse of the right-angled triangle  $TSV$ ;

$$\therefore TV = 2TP = 2PS,$$

$$\therefore QQ' = 2VT = 4PS.$$

**Def.** The focal chord parallel to the tangent at  $P$  is called the **parameter** of the diameter through  $P$ .

**45. Prop. XVII.** *The ordinate to any diameter is a mean proportional to the abscissa and the parameter of that diameter.* [ $QV^2 = 4SP \cdot PV$ .]



Let the tangent at  $Q$  meet the diameter through  $P$  in  $O$  and the axis in  $T$ .

Let the tangents at  $P$  and  $Q$  meet in  $K$ . Join  $SP$ ,  $SK$ ,  $SQ$ .

Then  $PK$  bisects the angle  $OPS$ ; [Prop. VI.]

$$\therefore \angle SPK = \angle KPO.$$

Since  $KP$ ,  $KQ$  are two tangents,

$$\angle SKP = \angle SQK \quad [\text{Prop. XI.}]$$

$$= \angle STQ \quad [\text{Prop. VII.}]$$

$$= \angle POK.$$

Hence the triangles  $SPK$ ,  $KPO$  are similar;

$$\therefore SP : PK = PK : OP;$$

$$\therefore PK^2 = SP \cdot OP = SP \cdot PV.$$

But, since  $OV = 2OP$ ,  $QV = 2PK$ ,

$$\text{hence } QV^2 = 4PK^2 = 4SP \cdot PV.$$

**Cor.** If  $QD$  be drawn perpendicular to the diameter  $PV$  of a parabola, and  $QV$  be the ordinate to that diameter; then will  $QD^2 = 4AS \cdot PV$ .

For, from similar triangles,

$$\begin{aligned} QD^2 &= QV^2 = PN^2 : PT^2 \quad [\text{Fig. Prop. IX.}] \\ &= SY^2 : SP^2 \\ &= SA : SP. \end{aligned}$$

From the relation  $QV^2 = 4SP \cdot PV$  it follows that if we have any two fixed straight lines  $OA$ ,  $OB$ , and if from a point  $Q$  the line  $QV$  be drawn parallel to  $OB$  to meet  $OA$  in  $V$ ; then, if  $Q$  move in such a manner that  $QV^2$  varies as  $OV$ , the point  $Q$  will describe a parabola of which  $OA$  is a diameter and  $OB$  the tangent at the extremity of that diameter.

If  $QV^2$  varies as  $OV$  it at once follows that the perpendicular from  $Q$  on  $OB$  varies as the square of the perpendicular on  $OA$ . Thus a point describes a parabola when it moves in such a manner that its perpendicular distance from one of two given straight lines (inclined at any angle) varies as the square of its perpendicular distance from the other.

It is also important to notice that, if a straight line cuts two fixed straight lines so that the intercept on one of the fixed lines, measured from their intersection, varies as the square of the intercept on the other; then the line will always touch a fixed parabola of which the fixed lines are respectively a diameter and the tangent at its extremity.

For, if  $OP$ ,  $OK$  are the fixed straight lines, and the line  $PK$  be such that  $OK^2 = OP \cdot OL$ , where  $OL$  is a fixed length; and, if through  $O$  a line be drawn making with  $KO$  an angle equal to the angle  $KOP$ , and on this line a point  $S$  be taken such that  $OS = OL$ . Then  $PK$  will always touch the parabola whose focus is  $S$ , which passes through  $O$ , and of which  $PO$  is a diameter.

**Ex. 1.** If  $QV$  be any ordinate to the diameter through a fixed point  $P$  on a parabola, and the parallelogram  $PVQR$  be completed, the line  $VR$  will be a tangent to a fixed parabola.

For  $PR^2 = QV^2 = 4SP \cdot PV$ .

Thus  $PR^2 \propto PV$ . Hence  $RV$  touches a fixed parabola of which  $VP$  is a diameter and  $PR$  is the tangent at its extremity.

**Ex. 2.**  $OA$ ,  $OB$  are two fixed straight lines, and any circle is drawn through the fixed point  $B$  to touch  $OA$ . Shew that, if the circle touch  $OA$  in the point  $P$ , and cut  $OB$  again in the point  $Q$ , the line  $PQ$  will always touch a fixed parabola.

Ex. 3. The algebraic sum of the ordinates, with reference to the axis, of the two extremities of any one of a system of parallel chords of a parabola is constant.

Ex. 4. If two chords of a parabola make equal angles with the axis, but are not parallel, the algebraic sum of the ordinates of their four extremities will be zero.

Ex. 5. Draw a focal chord of a parabola of given length.

Ex. 6. If two focal chords of a parabola be equal they must be equally inclined to the axis.

Ex. 7. The latus-rectum of a parabola is the shortest focal chord.

Ex. 8. The line joining the intersection of the tangents to the intersection of the normals at the extremities of any focal chord is parallel to the axis.

Ex. 9.  $QV$  is the ordinate to the diameter through  $P$ , and  $PW$  is the ordinate to the diameter through  $Q$ ; shew that  $VW$  is parallel to  $PQ$ .

Ex. 10.  $TQ$ ,  $TQ'$  are tangents to a parabola whose focus is  $S$ . The diameter through  $T$  cuts the directrix in  $K$ , and  $QQ'$  cuts the axis in  $O$ ; shew that  $SOKT$  is a parallelogram.

Ex. 11.  $OQ$ ,  $OQ'$  are two tangents to a parabola, and  $OM$  is drawn perpendicular to the axis; prove that  $QQ'$  cuts the axis in a point  $R$  such that  $MR$  is bisected by the vertex of the parabola.

Ex. 12.  $PQ$  is a chord of a normal at  $P$ , and the tangents at  $P$ ,  $Q$  meet in  $O$ . Shew that the diameter through  $O$  passes through the other extremity of the focal chord through  $P$ .

Ex. 13.  $TQ$ ,  $TQ'$  are tangents to a parabola whose focus is  $S$ , and the diameter through  $T$  cuts the curve in  $P$ ; shew that

$$SQ + SQ' = 2TP + 2PS.$$

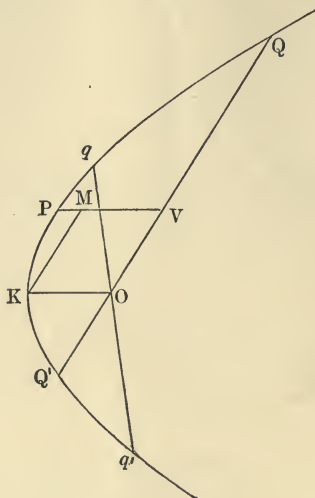
Ex. 14.  $QV$  is any ordinate to the fixed diameter  $PV$ ; shew that the middle point of  $QV$  describes a parabola.

Ex. 15.  $QQ'$  is any chord of a parabola parallel to the tangent at a fixed point  $P$ , and  $R$  is taken on  $QQ'$  such that  $QR : RQ'$  is constant. Shew that the locus of  $R$  is a parabola touching the given parabola at  $P$ .

Ex. 16. Given a parabola drawn on paper; find by means of a ruler and compasses its focus and directrix.



**46. Prop. XVIII.** *If two chords of a parabola intersect one another, the rectangles contained by their segments are in the ratio of the parallel focal chords.*



Let the chords  $QQ'$ ,  $qq'$  intersect in  $O$ .

Let  $V$  be the middle point of  $QQ'$ , and let  $PV$  be the diameter through  $V$ . Let  $OK$  be the diameter through  $O$ , and draw  $KM$  an ordinate to the diameter  $PV$ .

$$\begin{aligned}
 \text{Then } QO \cdot OQ' &= QV^2 - OV^2 = QV^2 - KM^2 \\
 &= 4SP \cdot PV - 4SP \cdot PM \\
 &= 4SP (PV - PM) \\
 &= 4SP \cdot KO.
 \end{aligned}$$

Similarly, if  $pv$  be the diameter which bisects  $qq'$ ,

$$qO \cdot Oq' = 4Sp \cdot KO.$$

$$\therefore QO \cdot OQ' : qO \cdot Oq' = 4SP : 4Sp.$$

But the focal chords parallel to the tangents at  $P$ ,  $p$  are  $4SP$ ,  $4Sp$  respectively. [Prop. XVI.]

If two chords of a parabola be drawn in fixed directions, the parallel focal chords will be the same for all positions of the point of intersection. Thus *the ratio of the rectangles contained by the segments of any two chords of a parabola is independent of the position of their point of intersection, and is therefore equal to the ratio of the squares of the parallel tangents.*

**47. Prop. XIX.** *If a circle cut a parabola in four points, the line joining any two of the points makes the same angle with the axis of the parabola as the line joining the other two points.*

For let  $K, L, M, N$  be the four points of intersection.

The chords  $KL$  and  $MN$  cannot be parallel unless they are perpendicular to the axis. For the line joining the middle points of parallel chords of a circle is perpendicular to the chords; the chords must therefore by Prop. XIII. be perpendicular to the axis of the parabola.

Let then  $KL, MN$  intersect in  $O$ .

Then, since the four points are on a parabola,

$$OK \cdot OL : OM \cdot ON = SP : SP', \quad [\text{Prop. XVIII.}]$$

where  $S$  is the focus and  $P, P'$  are the extremities of the diameters which bisect  $KL, MN$  respectively.

But, since the four points are on a circle,

$$OK \cdot OL = OM \cdot ON.$$

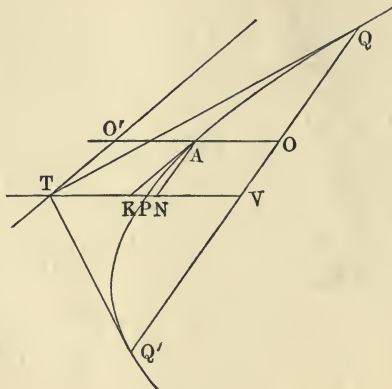
Hence  $SP$  and  $SP'$  are equal. They must therefore be equally inclined to the axis and on *opposite* sides of it, and the tangents at  $P, P'$  will therefore be inclined at equal angles to the axis, and these tangents are parallel respectively to the chords  $KL$  and  $MN$ .

In the same way it can be proved that  $KM$  and  $LN$ , and also that  $KN$  and  $LM$  are equally inclined to the axis.

If the two points  $K, L$  coincide, that is if the circle touch the parabola, the tangent at  $K$  and the chord  $MN$  will make equal angles with the axis. Also the lines  $KM, KN$  will make equal angles with the axis.

**Conversely**, the four extremities of any two chords of a parabola which are equally inclined to the axis, but are not parallel, lie on a circle.

**48\*. Prop. XX.** If any line be drawn through a fixed point to cut a parabola, the tangents at the points of intersection will meet on a fixed straight line.



Let any straight line through the fixed point  $O$  cut the parabola in  $Q, Q'$ . Let  $V$  be the middle point of  $QQ'$ . Then the tangents at  $Q, Q'$  meet at  $T$  on the diameter through  $V$ , and  $TP = PV$ .

Let  $OA O'$  be the diameter through  $O$ . Let the tangent at  $A$  cut  $TPV$  in  $K$ , and let  $AN$  be parallel to  $QQ'$ . Then  $KP = PN$ .

Hence

$$TK = TP - KP = PV - PN = NV = AO.$$

Hence the line through  $T$  parallel to the tangent at  $A$  will meet the diameter  $OA$  in a fixed point  $O'$  such that  $AO' = KT = OA$ .

\* The remainder of the chapter should be omitted on the first reading.

Thus for all directions of  $QQ'$  the point  $T$  is on a line parallel to the tangent at  $A$ , through the point  $O'$  such that  $AO' = OA$ .

In the figure the point  $O$  is within the parabola, but when the point  $O$  is outside the parabola the above proof applies without change, but in this case two lines can be drawn through  $O$  each of which will cut the parabola in coincident points, namely the tangents from  $O$  to the curve; and when the points  $Q, Q'$  coincide,  $T$  will coincide with them. Thus the locus of  $T$  will pass through the points of contact of the two tangents drawn from  $O$  to the parabola.

We can prove in a similar manner the converse of the above theorem, namely that *if any point be taken on a fixed straight line and tangents be drawn from it to the parabola, the line joining the points of contact of the tangents will pass through a fixed point.*

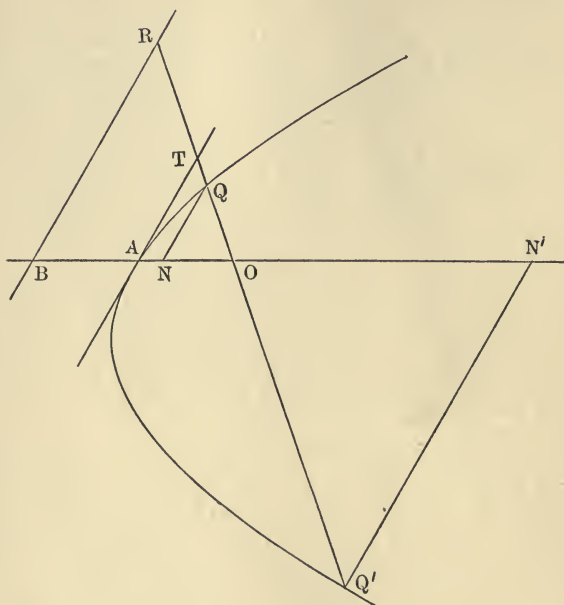
**Def.** The straight line which is the locus of the point of intersection of the tangents at the extremities of any chord through a fixed point, is called the **polar** of the fixed point, and the point is called the **pole** of the line with respect to the curve.

From Prop. V. we see that *the directrix is the polar of the focus, and that the focus is the pole of the directrix.*

**49. Prop. XXI.** *If any chord of a parabola be drawn through the fixed point  $O$ , cutting the curve in  $Q, Q'$ , the polar of  $O$  in the point  $R$ , and the tangent at  $A$ , the extremity of the diameter through  $O$ , in the point  $T$ ; then will  $TO$  be a mean proportional between  $TQ$  and  $TQ'$ , and  $TO$  will be the harmonic mean between  $RQ$  and  $RQ'$ .*

By Prop. XVIII., the ratio of  $TQ \cdot TQ'$  to  $TA^2$  is equal to the ratio of the squares of the parallel tangents. And, by Prop. XV. Cor., the ratio of the tangents parallel to  $TO$  and  $TA$  is equal to the ratio of  $TO$  to  $TA$ .

Hence  $TQ \cdot TQ' : TA^2 = TO^2 : TA^2$ ;  
 $\therefore TQ \cdot TQ' = TO^2 \dots\dots\dots(i).$



Let the diameter through  $O$  cut the polar of  $O$  in the point  $B$ ; then, since the polar of  $O$  is parallel to  $AT$ ,

$$OT : TR = OA : AB.$$

Hence, as  $OA = AB$ ,  $OT = TR$ .

But, by (i),  $TQ : TO = TO : TQ'$ ;

$$\therefore TQ + TO : TO - TQ = TO + TQ' : TQ' - TO.$$

That is, since  $TO = RT$ ,

$$RQ : QO = RQ' : OQ';$$

$$\therefore RQ : RQ' = QO : OQ' \\ = RO - RQ : RQ' - RO \dots(ii).$$

Hence  $RQ, RO, RQ'$  are in harmonic progression.

Again, if  $QN$ ,  $Q'N'$  be ordinates to the diameter  $AO$ , and therefore parallel to  $AT$ , we have

$$AN : AO = TQ : TO,$$

and  $AO : AN' = TO : TQ'.$

Hence, from (i),

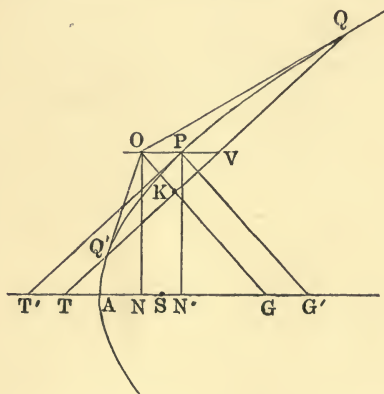
$$AN : AO = AO : AN',$$

or  $AN \cdot AN' = AO^2 \dots\dots\dots(iii),$

a result which is often useful.

50. Propositions VII. and VIII. may be generalised as follows.

*From any point  $O$  draw  $ON$  perpendicular to the axis of a parabola, and  $OKG$  perpendicular to the polar of  $O$  with respect to the parabola, meeting the polar in  $K$  and the axis in  $G$ , and let the polar of  $O$  meet the axis in  $T$ . Then will  $TA = AN$ ,  $ST = SG = SK$  and  $NG = 2AS$ .*



Let the diameter through  $O$  cut the curve in  $P$  and the polar in  $V$ . Let the tangent and normal at  $P$  cut the axis in  $T'$ ,  $G'$  respectively, and let  $PN'$  be the ordinate of  $P$ .

Then  $T'T = PV = OP = NN'$ .

But  $T'A = AN'$ ;

$$\therefore T'A - T'T = AN' - NN';$$

$$\therefore TA = AN.$$

Again  $GG' = OP = PV = T'T$ ,

and  $T'S = SG'$ ,

$$\therefore T'S - T'T = SG' - GG';$$

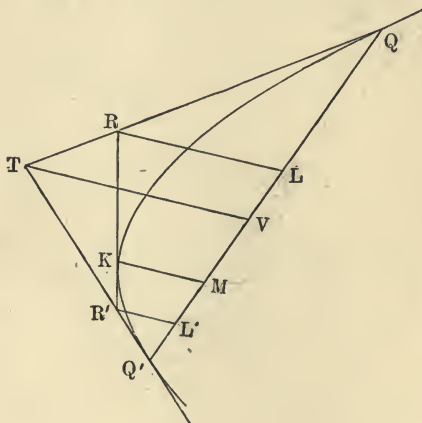
$$\therefore TS = SG$$

$= SK$ , since  $TKG$  is a right angle.

Also, the triangles  $ONG$ ,  $PN'G'$  are equal in all respects;

$$\therefore NG = N'G' = 2AS.$$

**51. Prop. XXI.** *Any two tangents to a parabola are cut proportionally by any other tangent.*



Let  $TQ$ ,  $T'Q'$  be any two tangents to a parabola, and let them be cut by any other tangent in the points  $R$ ,  $R'$  respectively; then we have to prove that

$$QR : RT = TR' : R'Q'.$$

Let  $K$  be the point of contact of  $RR'$ . Draw through  $R, R', T, K$  diameters cutting  $QQ'$  in the points  $L, L', V, M$  respectively.

Then  $LM = \frac{1}{2}QM$ , and  $ML' = \frac{1}{2}MQ'$ ;

$$\therefore LL' = \frac{1}{2}QQ' = QV.$$

Hence  $QL = VL'$ ,

and therefore also  $LV = L'Q'$ .

$$\begin{aligned}\text{Then } QR : RT &= QL : LV \\ &= VL' : L'Q' \\ &= TR' : R'Q' .\end{aligned}$$

We have also

$$\begin{aligned}RK : KR' &= LM : ML' = QL : LV \\ &= QR : RT .\end{aligned}$$

**Conversely.** If two fixed lines  $TQ, TQ'$  be cut by a moving line in  $R, R'$  respectively so that

$$QR : RT = TR' : R'Q' ,$$

then will the moving line  $RR'$  always touch the parabola which touches  $TQ, TQ'$  at  $Q$  and  $Q'$ .

**Ex. 1.** Two tangents  $TQ, TQ'$  to a parabola are cut by any third tangent in  $R, R'$  respectively. Shew that, if the parallelogram  $RTR'O$  be completed,  $O$  will be on  $QQ'$ .

Draw  $RO$  parallel to  $TQ'$  to meet  $QQ'$  in  $O$ . Then

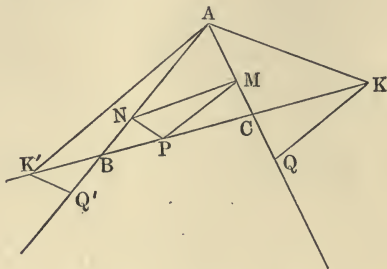
$$QO : OQ' = QR : RT = TR' : R'Q' ,$$

whence it follows that  $R'O$  is parallel to  $TQ$ .

It follows from this that if any parabola be drawn to touch three fixed straight lines, the chord of contact with any two will pass through a fixed point, namely through the other vertex of the parallelogram of which the two lines are sides and the third side the diagonal.

Ex. 2. From any point  $P$  on the base  $BC$  of a triangle  $ABC$  lines  $PN$ ,  $PM$  are drawn in fixed directions to cut  $AB$ ,  $AC$  respectively in  $N$ ,  $M$ . Shew that  $MN$  envelops a parabola.

[When  $PM$ ,  $PN$  are drawn parallel to  $AB$ ,  $AC$  respectively we have the converse of Ex. 1.]



We first notice that one position of  $MN$  will coincide with  $AB$ , namely when  $M$  is at  $A$ ; so also one position will coincide with  $AC$ . Hence we have to prove that  $MN$  touches a parabola which touches  $AB$  and  $AC$ . This could be at once tested provided the points of contact of  $AB$  and  $AC$  were known. These points of contact are found by the following constructions. Draw  $AK$  parallel to  $PN$  meeting  $BC$  in  $K$ , and  $KQ$  parallel to  $PM$  meeting  $AC$  in  $Q$ . Then draw  $AK'$  parallel to  $PM$  and  $K'Q'$  parallel to  $PN$ .

Now, since  $Q'K'$ ,  $PN$  and  $AK$  are parallel,

$$Q'N : NA = K'P : PK.$$

And, since  $K'A$ ,  $PM$  and  $KQ$  are parallel,

$$K'P : PK = AM : MQ.$$

Hence

$$Q'N : NA = AM : MQ.$$

This shews that  $MN$  touches the parabola which touches  $AQ$ ,  $AQ'$  at the points  $Q$ ,  $Q'$ .

**Or thus.** Let the circle  $NPM$  cut  $BC$  in  $L$ . Then

$$\angle LNM = \angle LPM = \text{constant angle},$$

and

$$\angle LMN = \angle NPB = \text{constant}.$$

Hence the triangle  $LMN$  is of *invariable form*. But it is easy to prove that if a triangle  $LMN$  of invariable form be inscribed in a given triangle  $ABC$ , the circles  $AMN$ ,  $BNL$ ,  $CLM$  will all meet in a point  $S$ , and that this point  $S$  is *fixed* for all positions of  $LMN$ . This shews that  $MN$  touches the parabola whose focus is  $S$  and which touches  $AB$ ,  $AC$ .

The other sides  $NL$ ,  $LM$  also touch fixed parabolas of which  $S$  is the focus.

# CONFOCAL AND COAXIAL PARABOLAS.

**52.** Parabolas which have the same focus and their axes in the same straight line have some interesting properties which we proceed to investigate.

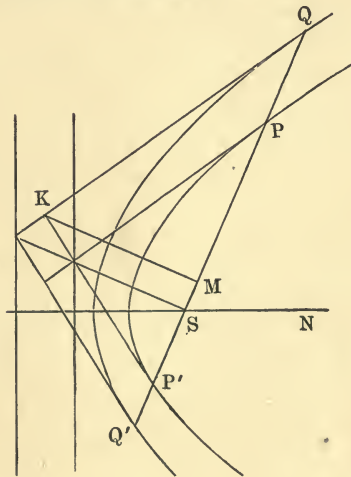
I. *Through any point two parabolas will pass having their concavities in opposite directions, and the two parabolas will intersect at right angles.*

For, if  $S$  be the focus and  $P$  any point, and a line be drawn through  $P$  parallel to the axis, and points  $M, M'$  on different sides of  $P$  be taken on this line so that  $MP = PM' = PS$ ; it is clear that parabolas with  $S$  as focus and lines perpendicular to the axis through  $M, M'$  respectively for the directrices will pass through  $P$ . The tangents at  $P$  to the two parabolas will bisect the angles  $SPM, SPM'$  respectively and will therefore be at right angles.

II. *One parabola can be drawn to touch any given straight line.*

Draw  $SY$  perpendicular to the given straight line, and produce it to  $M$  so that  $SY=YM$ . Through  $M$  draw a line parallel to the axis cutting the given straight line in  $P$ . Then it is clear that the parabola whose focus is  $S$  and whose directrix passes through  $M$  will touch the given straight line at  $P$ .

III. If a tangent to one parabola of the system be perpendicular to a tangent to another, their point of intersection will be on a straight line midway between their directrices.



Let  $QPSP'Q'$  be a common focal chord of two confocal and coaxial parabolas. The tangents at  $P, P'$  intersect at right angles on one directrix, and the tangents at  $Q, Q'$  intersect at right angles on the other directrix.

The tangents at  $P$  and  $Q$  will be parallel since each tangent makes with the axis an angle equal to half the angle  $PSN$ . Similarly the tangents at  $P'$  and  $Q'$  will be parallel. Hence the four tangents are along the sides of a rectangle. If  $K$  be the point of intersection of the tangents at  $Q, P'$ , and  $KM$  be drawn perpendicular to  $QSQ'$ ; then the distance of  $K$  from each directrix will be equal to  $SM$ . Hence  $K$  is on a straight line midway between the directrices.

[When the parabolas have their concavities in opposite directions, the perpendicular tangents to the two parabolas have their points of contact on the *same* side of the axis.]

IV. *If from any point  $T$  of a parabola two tangents be drawn to a confocal coaxial parabola, these tangents will be equally inclined to the tangent at  $T$ .*

For if, in the figure to Prop. XI.,  $TL$  be drawn parallel to the axis; then

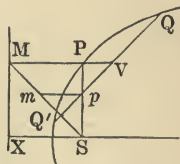
$$\angle PTS = \angle SPT = \angle PTL.$$

Thus  $TP, TP$  make equal angles with  $TS$  and  $TL$  respectively.

Also the tangent at  $T$  to a confocal coaxial parabola through  $T$  makes equal angles with  $TS$  and  $TL$ , whence it follows that  $TP, TP'$  make equal angles with the tangent at  $T$  to a confocal parabola which passes through  $T$ .

V. *If a chord of a parabola vary as the parallel focal chord it will touch a fixed confocal parabola.*

Let  $QQ'$  be a chord of a parabola, and  $PV$  the diameter which bisects the chord in  $V$ .



Then  $QV^2 = 4SP \cdot PV$ . Hence, if  $QV \propto 4SP$ , it follows that  $PV$  will vary as  $SP$ .

Let  $SP$  cut  $QQ'$  in  $p$ . Let  $PM$  be the perpendicular on the directrix. Join  $SM$ , and draw  $pm$  parallel to  $PM$  cutting  $SM$  in  $m$ .

Then, since  $PV$  and  $SP$  make equal angles with the tangent at  $P$ , they will make equal angles with  $QQ'$ . Hence  $PV = Pp$ .

Therefore, as  $PV$  varies as  $SP$ , it follows that  $Sp : SP$  is constant. Now  $Sp : pm = SP : PM$ ;  $\therefore Sp = pm$ . And, since  $Sm : SM$  is constant, it follows that  $m$  is on a fixed straight line parallel to  $MX$ . The locus of  $p$  is therefore a parabola whose focus is  $S$ , and since  $QQ'$  bisects the angle between  $Sp$  and  $pm$ , it follows that  $QQ'$  touches this fixed parabola.

VI. *The locus of the poles of a given straight line with respect to all parabolas of a confocal system is a straight line perpendicular to the given straight line.*

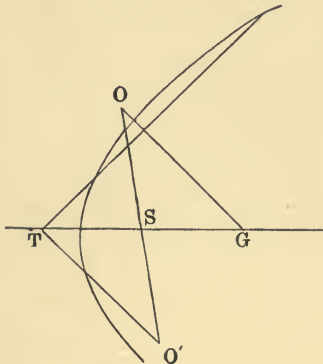
If the given straight line cut the axis in  $T$ , and  $G$  be taken on the axis such that  $TS=SG$ . Then we have seen (in Art. 47) that the pole of the given line with respect to any parabola whose focus is  $S$  and axis  $TSG$ , is on the line through  $G$  perpendicular to the given line.

VII. *The envelope of the polar of a given point with respect to all parabolas of the system is another parabola.*

Let  $O$  be the fixed point, and let its polar with respect to any parabola whose focus is  $S$  cut the axis in  $T$ , and let the perpendicular from  $O$  on its polar cut the axis in  $G$ .

Then

$$TS=SG.$$



Hence, if through  $T$  a line be drawn perpendicular to the polar, and therefore parallel to  $OG$ , it will cut  $OS$  produced in a point  $O'$  such that  $OS=SO'$ .

Hence the perpendicular from the *fixed point*  $O'$  on the polar meets it on a fixed line, namely on  $TSG$  the common axis of the system of parabolas.

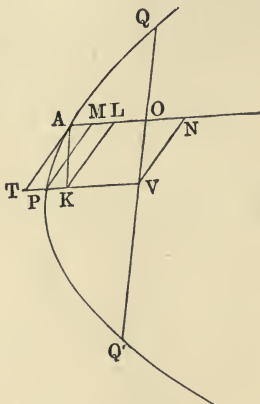
Therefore the polar envelops a parabola whose focus is  $O'$  and of which the axis of the system of parabolas is the tangent at the vertex.

53. The following are important examples.

Ex. 1. *The locus of the middle points of chords of a parabola which pass through a fixed point is another parabola.*

Let  $O$  be the fixed point,  $QQ'$  any chord through  $O$ . Let  $AO$  be the diameter through  $O$ .

Draw the diameter through  $V$  the middle point of  $QQ'$  cutting the curve in  $P$  and the tangent at  $A$  in  $T$ . Draw  $AK$  parallel to the tangent at  $P$  cutting  $PV$  in  $K$ ; and draw  $PM$ ,  $KL$ ,  $VN$  parallel to the tangent at  $A$ .



Then

$$AL = TK = 2TP = 2AM.$$

Also, since  $AK$  is parallel to  $OV$ , and  $TV$  parallel to  $AN$ ,  $AL = ON$ .

Then

$$\begin{aligned} VN^2 &= PM^2 = 4SA \cdot AM = 2SA \cdot AL \\ &= 2SA \cdot ON. \end{aligned}$$

Hence the locus of  $V$  is a parabola of which  $ON$  is a diameter and the tangent at  $O$ , the extremity of that diameter, is parallel to the tangent at  $A$ .

Ex. 2. Find the focus and directrix of a parabola which touches four given straight lines.

Draw the circum-circles of any two of the triangles formed by taking three of the four given lines, and let  $S$  be the point of intersection of these circles which is not on one of the given lines. Then the feet of the perpendiculars from  $S$  on the four given lines will lie in a straight line, and  $S$  will be the only point for which this is true.

Describe a parabola with  $S$  as focus, and with the line through the feet of the perpendiculars from  $S$  on the given straight lines as the tangent at the vertex; then this parabola will touch the four given straight lines.

Ex. 3. Find the focus and directrix of a parabola which passes through four given points.

Let  $A, B, C, D$  be the four given points, and let  $AC, BD$  intersect in  $O$ .

Take two points  $X, Y$  on  $AC, BD$  respectively such that

$$OX^2 : OY^2 = AO \cdot OC : BO \cdot OD.$$

Then [Prop. XV. Cor.]  $XY$  will be parallel to the axis of a parabola through the four given points. Also, if  $X', Y'$  be taken on  $AC, BD$  respectively so that  $X'O = OX$  and  $Y'O = OY$ , we see that there are two possible directions of the axis of the parabola which are parallel to the lines  $XY, X'Y'$  respectively. [These lines will be at right angles to one another if the four given points lie on a circle. This follows at once from Art. 44.]

Let  $V$  be the middle point of  $AD$ . Draw through  $V$  a line parallel to either of the possible axes, and let the line through  $B$  parallel to  $AD$  cut this diameter in the point  $W$ . Then, since  $AV$  and  $BW$  are ordinates to the diameter  $VW$ ,

$$AV^2 : BW^2 = VP : WP,$$

where  $P$  is the extremity of the diameter  $VW$ . Thus  $P$  can be easily found.

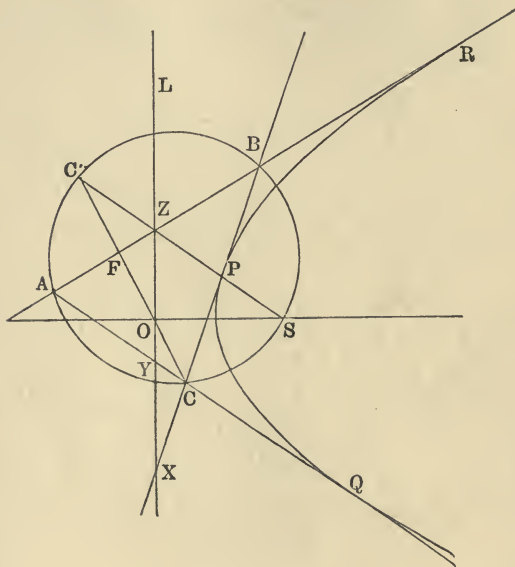
Then the tangent at  $P$  is parallel to  $AD$ , and the focus  $S$  lies on the line through  $P$  such that the tangent bisects the angle  $SPV$ .

We can find in a similar manner the extremity of the diameter which bisects another side, and also another straight line on which the focus must lie.

The focus is thus determined; and, knowing the focus, the direction of the axis, and one point, the directrix is at once found.

Ex. 4. *If the sides of a triangle touch a parabola, the orthocentre of the triangle will be on the directrix of the parabola.*

Let  $ABC$  be the triangle whose sides touch a parabola whose focus is  $S$  in  $P, Q, R$ . Let the directrix cut the sides  $BC, CA, AB$  respectively in  $X, Y, Z$ . Join  $SZ$ , and let  $SZ$  cut the circle  $ABCS$  in  $C'$ . Join  $CC'$  cutting the directrix in  $O$  and  $AB$  in  $F$ .



Then  $\angle CC'S = \angle CAS$  in the same segment  
 $= \angle SRA$   
 $= \text{complement of } \angle SZR$   
 $= \text{complement of } \angle FZC'.$

$\therefore CC'$  is perpendicular to  $AB$ .

But  $BA$  bisects the angle  $SZL$  or  $OZC'$ .

Hence

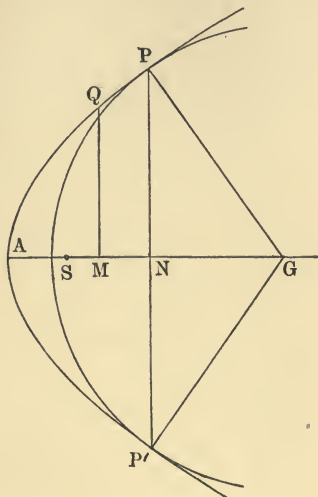
$$OF = FC';$$

$\therefore \angle OAB = \angle FAC' = \angle BAC' = \angle BCF = \text{complement of } \angle CBA;$

$\therefore OA$  is perpendicular to  $BC$ .

Hence  $O$  is the orthocentre, and it is a point on the directrix.

Ex. 5. If a circle touch a parabola at the extremities of a double ordinate  $PP'$ , the tangent drawn to the circle from any point on the parabola is equal to the perpendicular distance of that point from the chord of contact.



The normals at  $P, P'$  meet on the axis at  $G$ ; and the circle with centre  $G$  and radius  $GP$  will touch the parabola at  $P$  and  $P'$ .

If  $Q$  be any point on the curve, and  $QM$  be the ordinate of  $Q$ , then

$$\begin{aligned}
 QG^2 &= QM^2 + MG^2 \\
 &= QM^2 + (MN + 2AS)^2 \\
 &= 4AS \cdot AM + 4AS^2 + 4AS \cdot MN + MN^2 \\
 &= 4AS (AM + MN) + 4AS^2 + MN^2 \\
 &= PN^2 + NG^2 + MN^2 \\
 &= PG^2 + MN^2.
 \end{aligned}$$

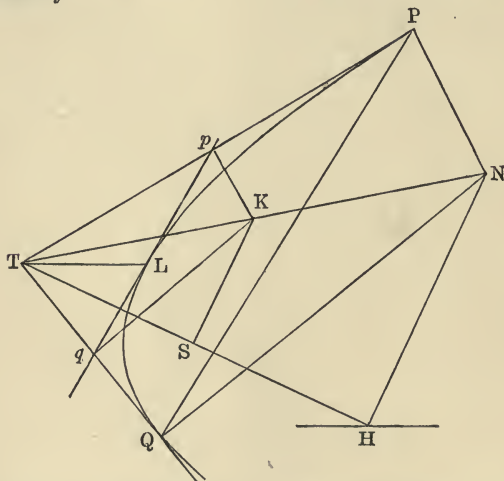
But  $QG^2 - PG^2 =$  square of tangent from  $Q$  to the circle whose centre is  $G$  and radius  $GP$ .

Hence square of tangent  $= MN^2$ .

Ex. 6. *The normals at the extremities of any one of a system of parallel chords of a parabola meet on a fixed straight line; also if the normals at three points on a parabola are concurrent, the circle through the three points will pass through the vertex of the parabola.*

Let  $PQ$  be any one of the system of parallel chords, and let the tangents and the normals at  $P$ ,  $Q$  intersect in the points  $T$ ,  $N$  respectively.

Let  $L$  be the extremity of the diameter through  $T$ , and let the tangent at  $L$  cut  $TP$ ,  $TQ$  respectively in the points  $p$ ,  $q$ . Then, since the direction of  $PQ$  is fixed,  $L$  is a fixed point and  $TL$  is a fixed diameter. Also  $p$ ,  $q$  are the middle points of  $TP$ ,  $TQ$  respectively.



Draw  $pK$ ,  $qK$  perpendicular to  $Tp$ ,  $Tq$  respectively. Then, since  $p$ ,  $q$  are the middle points of  $TP$ ,  $TQ$  respectively, it follows that  $TKN$  is a straight line and that  $TK=KN$ .

Since  $TpK$  and  $TqK$  are right angles,  $TK$  is a diameter of the circle  $Tpq$ . But this circle passes through  $S$ , the focus of the parabola.

Hence  $TSK$  is a right angle.

Also  $\angle SKT = \angle SpT = \angle SpL + \angle LPT = \angle SpL + \angle LSp = \angle SLq$ .

But  $SLq$  is a constant angle, since  $L$  is a fixed point.

Hence  $SKT$  is a constant angle.

Therefore the right-angled triangle  $KST$  is of invariable form, and therefore the ratio  $SK : ST$  is constant.

Now produce  $TS$  to  $H$  so that  $SH=TS$ .

Then  $\angle THN = \angle TSK = \text{right angle}$ .

Also  $TH : HN = 2TS : 2SK = \text{const.}$

Hence  $SH : HN$  is constant.

But, since  $S$  is a fixed point such that  $TS=SH$ , and the locus of  $T$  is a fixed diameter, the locus of  $H$  must also be a fixed diameter.

Then, since  $SHN$  is a right angle and  $SH : HN$  is constant, the triangle  $SHN$  is of invariable form, so that the angle  $HSN$  is constant and  $SN : SH$  is constant. The locus of  $N$  is therefore similar to the locus of  $H$ , and we have just seen that  $H$  moves on a fixed straight line. Hence the locus of  $N$  is also a straight line.

Thus the locus of the point of intersection of the normals at the extremities of any one of a system of parallel chords of a parabola is a straight line. [The above proof is given in Taylor's *Geometry of Conics*, p. 224.]

Now let  $A$  be the vertex of the parabola, and let  $AR'$  be parallel to the system of chords, and let  $R$  be the other extremity of the ordinate through  $R$ , so that  $AR$  and  $AR'$  are equally inclined to the axis.

Then, if the normal at  $R'$  meet the axis in  $G$ ,  $G$  is one point on the straight line on which all the intersections of normals lie. But  $G$  is also on the normal at  $R$ ; moreover, if one of the parallel chords be drawn through  $R$ , the normals at its extremities must intersect on the normal at  $R$ . Hence the normal at  $R$  contains two of the points on the straight line on which all the intersections of normals lie; and therefore all the pairs of normals must intersect on the normal at  $R$ . Hence the normals at  $P$ ,  $Q$  meet at a point on the normal at  $R$  which is such that  $PQ$  and  $AR$  are equally inclined to the axis of the parabola, whence it follows that  $P$ ,  $Q$ ,  $R$ ,  $A$  are cyclic.

This proves Harvey's Theorem that if the normals at three points on a parabola be concurrent the circle through the three points will pass through the vertex of the parabola.



## EXAMPLES ON THE PARABOLA.

1. Shew that, if the length of a chord of a parabola be double the distance of its middle point from the directrix, the chord must pass through the focus.

2. On a given base  $AB$  any isosceles triangle  $ABP$  is described, and on  $AP$  as base another isosceles triangle  $APQ$  similar to the former is described. Shew that the locus of  $Q$  is a parabola whose focus is  $A$  and whose directrix bisects  $AB$  at right angles.

3.  $A$  is a fixed point,  $Q$  is any point on a fixed straight line,  $QP$  is drawn perpendicular to the fixed line, and  $AP$  is perpendicular to  $AQ$ . Shew that the locus of  $P$  is a parabola.

4.  $PSP'$  is a focal chord of a parabola, and the lines through  $P, P'$  parallel to the axis meet the normals at  $P', P$  respectively in  $K, K'$ . Shew that  $PP'K'K$  is a rhombus.

5. A line through the vertex of a parabola perpendicular to the tangent at any point  $P$  meets the line through  $P$  parallel to the axis in  $Q$ . Shew that the locus of  $Q$  is a straight line perpendicular to the axis.

6. The normal at the point  $P$  of a parabola cuts the axis in  $G$ , and  $SY$  is the perpendicular from the focus on the tangent at  $P$ .  $YS$  is produced to  $R$  so that  $YS=SR$ . Shew that  $PYRG$  is a rectangle, and that the circle  $PYRG$  passes through the vertex of the parabola.

7. Shew that the length of the normal  $PG$  at any point  $P$  of a parabola is equal to the ordinate which bisects  $PG$ .

8.  $A$  is the vertex of a parabola and  $P$  is any point on the curve. A line through  $P$  perpendicular to  $PA$  cuts the axis in  $G$ , and  $Q$  is taken on  $GP$  produced so that  $GP=PQ$ . Shew that the locus of  $Q$  is a parabola whose focus is  $A$ .

9. Shew that, if two parabolas have a common directrix, the line joining their common points is parallel to and midway between the line joining the points of contact of their common tangents.

10. A circle is drawn to touch the axis of a parabola, the focal distance  $SP$  of any point  $P$ , and also the diameter through  $P$ ; shew that the centre of the circle must lie on another parabola or on the tangent at the vertex of the original parabola.

11.  $ACP$  is a sector of a given circle whose centre is  $C$ , and of which the radius  $CA$  is fixed. A circle is drawn touching the arc  $AP$  externally and also touching  $CA, CP$  produced. Shew that the centre of this circle, for different positions of  $P$ , is always on one or other of two parabolas.

12. Shew that the locus of the point of intersection of the normals to a parabola at the extremities of a focal chord is another parabola.

13. An endless string  $OPQ$  is fastened at  $O$  and two small beads  $P$ ,  $Q$  slide along it; the string is kept stretched and the beads move so that  $OP$  is always equal to  $OQ$  and  $PQ$  always fixed in direction; shew that  $P$  and  $Q$  move on arcs of two parabolas with a common focus and latus-rectum.

14.  $P$  is any point on a parabola whose focus is  $S$  and vertex  $A$ . The perpendicular from  $S$  to  $AP$  cuts the tangent at the vertex in  $R$ . Shew that the ordinate of  $P$  is  $4AR$ .

15. If perpendiculars be let fall on any tangent to a parabola from two fixed points on the axis equidistant from the focus, the difference of their squares will be constant.

16. Through any point  $P$  of a parabola the chord is drawn which makes an angle with the axis equal to that made by the tangent at  $P$ . Shew that the chord touches a fixed parabola.

17.  $PN$  is the ordinate at any point  $P$  of a parabola, and the tangent at  $P$  cuts the tangent at the vertex in  $Y$ . Shew that  $NY$  touches an equal parabola.

18. From an external point  $O$  two tangents are drawn to a parabola, and from the points where they cut the directrix two other tangents are drawn meeting the tangents from  $O$  in  $A$ ,  $B$ . Prove that  $AB$  passes through the focus  $S$ , and is perpendicular to  $OS$ .

19. From a point  $P$  on a circle the ordinate  $PN$  is drawn to a fixed diameter  $AA'$  and is produced to  $Q$  so that the square on  $PN$  is equal to the rectangle contained by  $NQ$  and a given line; prove that the locus of  $Q$  is a parabola.

20.  $TP$ ,  $TQ$  are the tangents at the points  $P$ ,  $Q$  of a parabola whose focus is  $S$ ; prove that, if  $SP+SQ$  be constant, the locus of  $T$  is a parabola.

21.  $PSP'$  is any focal chord of a parabola whose vertex is  $A$ , and  $AP$ ,  $AP'$  cut the latus-rectum in  $K$ ,  $K'$  respectively. Shew that, if  $PN$ ,  $P'N'$  be the ordinates of  $P$ ,  $P'$  respectively, then  $NPSK'$  and  $N'P'SK$  will be parallelograms.

22.  $TQ$ ,  $TQ'$  are tangents to a parabola whose focus is  $S$ , and the diameter through  $T$  cuts the curve in  $P$ ; shew that

$$TQ \cdot TQ' = 4TP \cdot TS.$$

23. If a parabola roll upon an equal parabola which remains fixed, the vertices of the parabolas originally coinciding, the focus of the moving parabola describes the directrix of the fixed parabola, and the latus-rectum and the tangent at the vertex of the moving parabola touch fixed circles.

24. Two parabolas have a common focus and from any point on their common tangent are drawn the other tangents to the two parabolas. Prove that the angle between these tangents is equal to the angle between the axes of the conics.

25. Shew that, if  $TP$ ,  $TQ$  are tangents to a parabola,  $PQ$  being the normal at  $P$ ; then will  $TQ$  be bisected by the line through  $S$  perpendicular to  $TS$ .

26. Any number of equal parabolas have parallel axes and a common point; shew that their vertices lie on a parabola whose vertex is at the given point.

27. The latus-rectum of a parabola is given, also the direction of its axis and a fixed point on the curve. Shew that the locus of its focus is a parabola.

28. Shew that all parabolas which have a given directrix and a given tangent at a given point, will touch a fixed parabola of which the given point is focus.

29. Shew that all parabolas which have a common directrix, and whose foci are on a fixed circle, touch two fixed parabolas.

30. Prove that the locus of points at which two parabolas with the same focus subtend equal angles is the straight line bisecting the angle between their directrices.

31. If the triangle formed by three tangents to a parabola be isosceles, the line joining the vertex of the triangle to the focus will pass through the point of contact of the base.

32. If an equilateral triangle circumscribe a parabola, the lines from the focus to the angular points of the triangle will pass through the points of contact.

33. If a quadrilateral be inscribed in a circle one of its three diagonals will pass through the focus of the parabola which touches its sides.

34. Normals at  $P, p$  the extremities of a focal chord of a parabola intersect the axis in  $G, g$ . Shew that the line drawn through the middle point of  $Pp$  perpendicular to  $Pp$  will pass through the middle point of  $Gg$ .

35. Shew that two parabolas with parallel axes can only intersect in two points.

36. The normals at  $P, p$  the extremities of a focal chord of a parabola intersect the axis in  $G, g$  respectively, and the tangents at  $P, p$  meet in  $T$ . Shew that the circles  $SPG, Spg$  intersect at a point  $R$  on  $TS$  produced such that  $TS=SR$ .

37. If  $ABC$  be a triangle inscribed in a parabola, and  $A'B'C'$  be a triangle formed by three tangents parallel to the sides of the triangle  $ABC$ , then will the sides of  $ABC$  be four times the corresponding sides of  $A'B'C'$ .

38. If the tangent at a point  $P$  of a parabola meet the axis in  $T$ , and the chord  $PQ$  and the tangent  $PT$  make equal angles with the axis; then  $PQ=4PT$ .

39. If two tangents be drawn to a parabola, the perpendicular from the focus on their chord of contact will pass through the middle point of their intercept on the tangent at the vertex.

40. If the ordinate  $PN$  at any point  $P$  of a parabola be produced to  $Q$  such that  $PQ=SP$ ; prove that the locus of  $Q$  is a parabola which touches the tangent at the vertex of the original parabola, the corresponding diameter being the tangent at the extremity of the latus-rectum of the original parabola.

41. The tangent at a variable point  $P$  on a parabola meets the tangent at a fixed point  $Q$  in  $T$  and  $PT$  is divided in a fixed ratio in  $R$ . Prove that the locus of  $R$  is a parabola touching the given parabola at  $Q$ .

42. Shew that the envelope of the directrices of all parabolas which have a common vertex  $A$  and pass through a fixed point  $P$  is a parabola the length of whose latus-rectum is  $AP$ .

43. Inscribe in a given parabola a triangle having its sides parallel to three given straight lines.

44. If through a fixed point  $A$  a straight line be drawn meeting two fixed lines  $OD$ ,  $OE$  in  $B$  and  $C$  respectively, and on it a point  $P$  be taken such that  $AC \cdot AP = AB^2$ ; prove that the locus of  $P$  is a parabola which passes through  $A$  and  $O$ , and has its axis parallel to  $OD$ , and the tangent at  $A$  parallel to  $OE$ .

45. Inscribe a circle in a segment of a parabola cut off by a double ordinate.

46. A point  $V$  is taken on an ordinate  $PM$ , produced, of a parabola, and  $ME$  is taken on  $MP$  a mean proportional between  $PM$  and  $MV$ ; the diameters through  $E$ ,  $V$  meet the curve in  $R$ ,  $Q$  respectively. Prove that  $PQ$  meets the axis in the foot of the ordinate of  $R$ .

47. Focal chords of a parabola at right angles to one another meet the directrix in  $T$ ,  $T'$ . Shew that the bisectors of the angles between the tangents from either  $T$  or  $T'$  are parallel to the tangents from the other.

48.  $ABC$  is any isosceles triangle described on a given base  $AB$ , and the tangents at  $A$ ,  $C$  to the circle  $ABC$  intersect in  $P$ . Shew that the locus of  $P$  is a parabola whose focus is  $A$ , whose axis is along  $AB$  and whose latus-rectum is equal to  $AB$ .

49. If a leaf of a book be folded over so that one of the outer corners is on the inner side, the line of the crease will envelop a parabola whose directrix is the inner side.

50. Shew that a line which cuts two fixed lines  $OA$ ,  $OB$  in the points  $P$ ,  $Q$  respectively so that  $OP+OQ$  is constant, touches a fixed parabola.

51. Shew that a line which cuts two fixed straight lines so that the difference of the intercepts, measured from their point of intersection, is constant envelops a parabola.

52. In a given regular polygon is inscribed any regular polygon of the same number of sides; prove that the envelope of each side is a parabola.

53. A line cuts two given circles so that the intercepted chords are equal; shew that the line envelops a parabola.

54. Shew that all chords of a parabola, which have their middle points on a line perpendicular to the axis of the parabola, will touch another parabola.

55. If a chord of a parabola subtend a right angle at the vertex it will cut the axis at a distance from the vertex equal to the latus-rectum.

56.  $QQ'$  is any chord of a parabola; any diameter cuts the tangent at  $Q$  in  $T$ , the curve in  $P$  and the chord  $QQ'$  in  $K$ ; shew that

$$TP : PK = QK : KQ'.$$

57. Draw through a given point within a parabola a chord which shall be divided in a given ratio at the point.

58. The tangent at any point  $P$  of a parabola meets the tangents drawn from any point  $O$  in  $A$ ,  $B$  and the diameter through  $O$  in  $C$ . Prove that  $AP = CB$ .

59. Any chord  $PQ$  of a parabola cuts the axis in  $O$ ; shew that

$$PQ^2 = AP^2 + AQ^2 + 2RO^2 - 2AO^2,$$

$A$  being the vertex, and  $RO$  the ordinate through  $O$ .

60.  $T$  is the intersection of the tangents at the ends of any chord of a parabola,  $T'$  a corresponding point for any perpendicular chord; shew that the rectangle of the abscissæ of  $T$  and  $T'$  is equal to the rectangle of the intercepts of the chords on the tangent at the vertex.

61. Having given a point, a tangent and the direction of the axis of a parabola, shew that the locus of its focus is another parabola.

62. Two parabolas are described touching the sides of a triangle and having their foci at the extremities of a diameter of the circum-circle; shew that their axes intersect on the circum-circle and that the tangents at their vertices intersect on the nine-point circle of the triangle.

63. Through two fixed points  $A$ ,  $B$  on the axis of a parabola two chords  $PAQ$ ,  $PBR$  are drawn, and  $QR$  cuts the axis in  $T$ ; shew the ratio  $TQ : TR$  is independent of the position of  $P$ .

64. The locus of the centre of a circle which cuts a given straight line and a given circle so that the chords are equal and of constant length is a parabola.

65. The tangent at a point  $P$  of a parabola makes the same angle with the axis as the line drawn from the focus to the point of intersection of the tangents at two points  $Q$  and  $R$ . Shew that this relation is symmetrical and that the circum-circle of the triangle formed by the three tangents will touch the axis of the parabola.

66.  $PSP'$  is a focal chord of a parabola;  $V$  is the middle point of  $PP'$ , and  $VG$  is drawn perpendicular to  $PSP'$  to cut the axis in  $G$ . Shew that  $SG$ ,  $VG$  are the arithmetic and geometric means between  $SP$  and  $SP'$ .

67.  $OA$ ,  $OB$  are tangents to a parabola; shew that the circles passing through  $O$  and touching  $AB$  at  $A$  and  $B$  respectively will intersect on the diameter through  $O$ , and will have their centres on the directrix.

68. Any circle whose centre is at a given point cuts two fixed parallel straight lines in the points  $A, A'$  and  $B, B'$  respectively. Shew that the lines  $AB, AB', A'B, A'B'$  will all touch a fixed parabola.

69.  $P$  is any point on a parabola whose focus is  $S$ , and  $PS$  is produced to  $P'$  so that  $PS = SP'$ . Also  $P'Q, P'R$  are the tangents from  $P'$  to the parabola. Shew that the circle  $PQR$  touches the parabola at  $P$  and passes through  $P'$ .

70. Two parabolas have a common focus and their axes in opposite directions; prove that the locus of the middle points of chords of either which touch the other is another parabola.

71. The three middle points of the diagonals of any quadrilateral are on a straight line parallel to the axis of the parabola which touches the sides of the quadrilateral.

72. Two equal and similarly situated parabolas have the same axis, and a tangent is drawn at any point  $Q$  of the inner which cuts the outer in the points  $P, P'$ . Shew that  $Q$  is the middle point of  $PP'$ , and that the distance between the diameters through  $P$  and  $P'$  is constant for all positions of  $Q$ .

73. Two confocal and coaxial parabolas are cut by a line parallel to their axis in the points  $P, P'$ , and the tangents at  $P, P'$  intersect in  $T$ . Shew that  $T$  is midway between their directrices, and that  $TS$  bisects the exterior angle  $PSP'$ .

74. The tangents at two points one on each of two confocal and coaxial parabolas meet in  $T$ ; shew that if  $T$  be equidistant from the diameters through the points of contact of the tangents it will be also equidistant from the directrices of the parabolas.

75.  $PP', QQ'$  are the chords of contact of pairs of tangents from a point to two confocal and coaxial parabolas; shew that if  $P, Q, S$  are in a straight line, so also will  $P', Q', S$  be, and that  $PQ'$  and  $QP'$  will be parallel to the axis.

76. If a circle through the focus of a parabola touch the curve at  $P$  and cut it at  $L$  and  $M$ , and the circle cut the axis in  $N$ ; then will  $LP = MN$ .

77.  $PQ$  is a chord of a parabola normal at  $P$ ;  $QR$  is drawn parallel to the axis to meet the double ordinate  $PP'$  produced in  $R$ . Prove that the rectangle contained by  $PP'$  and  $P'R$  is constant.

78.  $TA, TA'$  are two tangents to a parabola and  $P$  is any other point on the curve; the tangent at  $P$  meets the diameters through  $A, A'$  in  $a, a'$  respectively; also lines through  $P$  parallel to the tangents at  $A, A'$  meet the diameters through  $A', A$  respectively in  $Q', Q$ . Shew that  $QQ'$  is parallel to the tangent at  $P$ , and that, if  $O$  be the pole of  $QQ'$ ,  $Oa, Oa'$  are parallel to the tangents at  $A', A$  respectively.

79. A triangle  $ABC$  is formed by three tangents to a parabola, another triangle  $DEF$  is formed by joining the points in which the chords through two points of contact meet the diameter through the third. Shew that  $A, B, C$  are the middle points of the sides of  $DEF$ .

80. A circle is described on a chord  $AB$  of a parabola as diameter, and cuts the curve again in  $C, D$ . Prove that, if the chord  $AB$  be fixed in direction, the difference of the squares of  $AB$  and  $CD$  is constant.

81. The tangent at any point  $P$  of a parabola is cut by any two other tangents in the points  $R, R'$ , and the diameter through  $P$  is cut by the chord of contact of those tangents in  $O$ ; shew that  $RP \cdot PR' = SP \cdot PO$ ,  $S$  being the focus.

82.  $TQ, TR$ , tangents to a parabola, meet the tangent at  $P$  in  $X$  and  $Y$  respectively, and  $TU$  is drawn parallel to the axis, meeting the parabola in  $U$ . Prove that the tangent at  $U$  passes through the middle point of  $XY$ , and that, if  $S$  be the focus,

$$XY^2 = 4SP \cdot TU.$$

83. Two fixed tangents are met by a variable tangent in the points  $X, Y$ . Shew that, if a chord of the parabola be drawn equal and parallel to  $XY$ , it will envelop an equal parabola.

84. If  $PQ$  be a focal chord of a parabola and  $R$  any point on the diameter through  $Q$ ; shew that the focal chord parallel to  $PR = \frac{PR^2}{PQ}$ .

85. Through any point on a parabola two chords are drawn equally inclined to the tangent there. Shew that their lengths are proportional to the portions of their diameters intercepted between them and the curve.

86.  $PP'$  is a focal chord of a parabola, and the normals at  $P, P'$  meet the curve again in  $Q, Q'$ . Shew that  $QQ'$  is parallel to  $PP'$ .

87. Shew that the locus of the focus of a parabola which touches two given straight lines and whose directrix passes through a given point, is a circle.

88.  $OQ, OQ'$  are two tangents to a parabola, and the diameter through  $O$  cuts the curve in  $P$  and  $QQ'$  in  $V$ ; shew that, if the tangent at  $P$  cut  $OQ, OQ'$  respectively in  $R, R'$ , then will  $QR', Q'R$  be divided by the parabola in the ratio 8 to 1.

89. Two circles are drawn each touching a parabola at the extremities of a double ordinate. Shew that the sum or the difference of the lengths of the tangents drawn to these circles from any point on the parabola is constant and equal to the distance between the chords of contact.

90. Two circles have double contact with a parabola; shew that their radical axis is midway between their chords of contact.

91.  $P, Q, R, S$  are any four points on a parabola;  $PQ$  meets the diameter through  $R$  in  $R'$  and  $RS$  meets the diameter through  $Q$  in  $Q'$ ; shew that  $Q'R'$  is parallel to  $PS$ .

92. The polar of the middle point of a normal chord of a parabola meets the focal vector to the point of intersection of the chord and the directrix on the normal at the further end of the chord.

93. The axes of the two parabolas which pass through four given points on a circle will intersect at right angles in the centroid of the four points.

94. Shew that, if two parabolas intersect in four points on a circle, their axes must be at right angles; and conversely, that if the axes of two parabolas be at right angles and they intersect in four points, the four points will be cyclic.

95. A variable tangent to a given parabola cuts a fixed tangent in the point  $P$ ; shew that the line through  $P$  perpendicular to the variable tangent envelops another parabola.

96. On any chord of a parabola as diameter a circle is described cutting the parabola in two other points. Shew that the portion of the axis intercepted between the original chord and the chord joining the other two points is equal to the latus-rectum.

97. If a triangle be inscribed in a parabola, the points where the sides produced meet the tangents at the opposite angles will lie on a straight line.

98. The ratio of the segments into which any tangent to a parabola is divided by three fixed tangents is constant.

99. If  $TP$ ,  $TQ$  be two tangents to a parabola, the perpendicular distance of  $T$  from any other tangent will be a mean proportional between the perpendicular distances of  $P$  and  $Q$  from the same tangent.

100. If four tangents be drawn to a parabola, and if  $A$  and  $A'$ ,  $B$  and  $B'$ ,  $C$  and  $C'$  be the extremities of the three diagonals of the quadrilateral formed by the tangents; then will the product of the perpendiculars from  $A$  and  $A'$  on any tangent, be equal to the product of the perpendiculars from  $B$  and  $B'$  or from  $C$  and  $C'$  on that tangent.

## CHAPTER III.

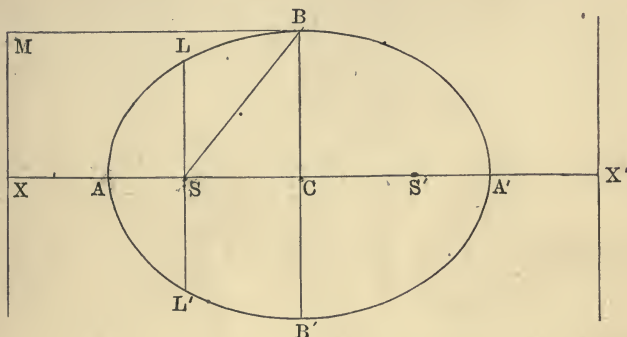
### THE ELLIPSE.

54. FROM the definition of an ellipse as the locus of a point which moves in the plane containing a given point, called the focus, and a given straight line, called the directrix, in such a manner that its distance from the focus is always in a constant ratio to its perpendicular distance from the directrix, we have already proved in Chapter I. that an ellipse is symmetrical about the line through its focus perpendicular to its directrix; and also that, if this line cut the curve in the points  $A$ ,  $A'$ , and  $C$  be the middle point of  $AA'$ , the ellipse is symmetrical about the line through  $C$  parallel to the directrix, whence it follows that there is a second focus in the line  $AA'$  and a second directrix perpendicular to  $AA'$ .

We have also proved that, if  $S$ ,  $S'$  be the two foci, and if  $AA'$  produced cut the directrices in the points  $X$ ,  $X'$  respectively, then

$$CS : CA = CA : CX = SA : AX.$$

The terminated lines  $AA'$  and  $BB'$  are called respectively the **major** and the **minor** axes of the ellipse.



Let  $BM$  be the perpendicular from  $B$  on the directrix.  
Then

$$SB : BM = SA : AX = CA : CX.$$

Hence, as  $BM = CX$ ,  $SB = CA$ .

$$\text{Therefore } BC^2 = BS^2 - CS^2 = CA^2 - CS^2.$$

$$\begin{aligned} \text{Now } CA^2 - CS^2 &= (CA - CS)(CA + CS) \\ &= AS \cdot SA', \text{ since } CA = CA'. \end{aligned}$$

$$\text{Also } CA^2 - CS^2 = CS \cdot CX - CS^2 = CS \cdot SX.$$

$$\text{Thus } \mathbf{BC^2 = CA^2 - CS^2 = AS \cdot SA' = CS \cdot SX.}$$

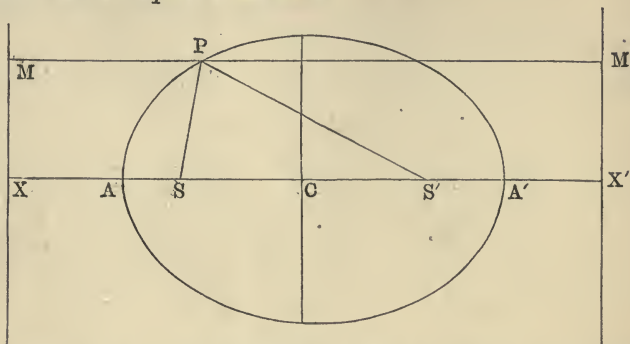
Again, if  $LSL'$  be the latus-rectum,

$$SL : SX = SA : AX = CS : CA ;$$

$$\therefore \mathbf{SL \cdot CA = CS \cdot SX = BC^2.}$$

Thus the semi-minor axis is a mean proportional to the semi-major axis and the semi-latus-rectum.

**55. Prop. I.** *The sum of the focal distances of any point on an ellipse is constant.*



Let  $S, S'$  be the foci and  $P$  be any point on the ellipse. Join  $SP, S'P$ , and draw through  $P$  the line  $MPM'$  perpendicular to and terminated by the directrices.

Then  $SP : MP = CA : CX$ ;

and  $S'P : PM' = CA : CX$ ;

$$\therefore SP + S'P : MP + PM' = CA : CX.$$

But  $MP + PM' = MM' = XX' = 2CX$ .

Hence  $SP + S'P = 2CA$ .

It can easily be proved that, if  $Q$  be any point external to the ellipse,  $SQ + S'Q$  is greater than  $2CA$ .

For, if  $SQ$  cut the ellipse in  $P$ ,

$$SQ + S'Q = SP + PQ + S'Q > SP + PS'.$$

Similarly, if  $Q$  be an internal point,  $SQ + S'Q < 2CA$ .

The above property of an ellipse enables us to describe the curve by the continuous motion of a point.

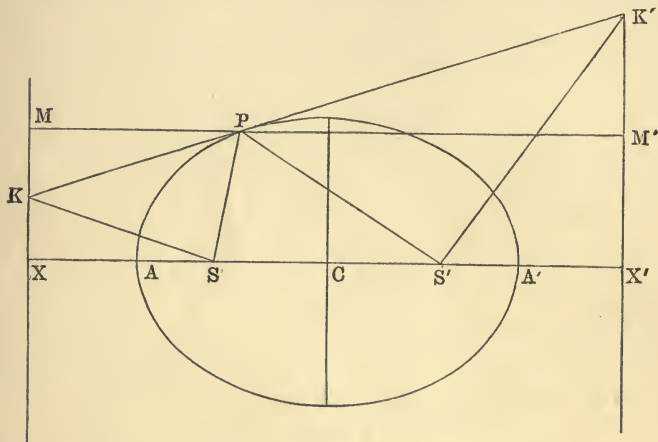
For, if the ends of a string of finite length be fastened at two points  $S, S'$ , and the string be kept tight by a pencil moving along it; then, if  $P$  be any position of the point of the pencil,  $SP + S'P$  will be constant and equal to the length of the string, and therefore  $P$  will be on an ellipse whose foci are  $S, S'$  and whose major axis is equal to the length of the string.

**56. Prop. II.** *The tangent at any point of an ellipse is equally inclined to the focal distances of the point.*

Let  $S, S'$  be the foci of the ellipse, and let  $P$  be any point on the curve.

Draw through  $P$  the line  $MPM'$  perpendicular to and terminated by the directrices.

Let the tangent at  $P$  cut the directrices in  $K, K'$ .



Join  $SP, S'P, SK$  and  $S'K'$ .

Then, from the similar triangles  $MPK, M'PK'$ ,

$$\begin{aligned} KP : PK' &= MP : PM' \\ &= SP : S'P. \end{aligned}$$

Also we know [Art. 12] that the angles  $KSP$  and  $K'S'P$  are right angles.

Hence the triangles  $KPS, K'PS'$  are similar, and therefore

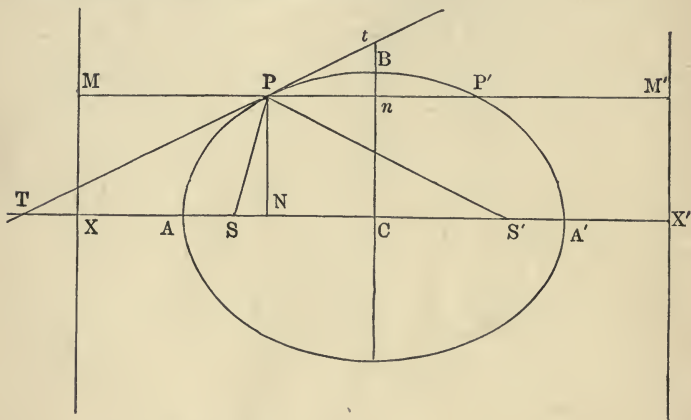
$$\angle SPK = \angle S'PK'.$$

Thus the tangent at  $P$  to the ellipse bisects the angle between  $SP$  and  $S'P$  produced.

Since the normal is perpendicular to the tangent, it follows that the normal bisects the angle  $SPS'$ .

**57. Prop. III.** *If the tangent at any point  $P$  of an ellipse cut the major axis  $CA$  produced in  $T$ , and  $PN$  be the perpendicular on the axis; then will  $CN \cdot CT = CA^2$ .*

Draw  $MPM'$  parallel to the axis-major cutting the directrices in the points  $M, M'$ .



Then, since  $TP$  bisects the exterior angle  $SPS'$ ,

$$\begin{aligned} ST : S'T &= SP : S'P \\ &= PM : M'P = NX : X'N; \end{aligned}$$

$$\therefore ST + S'T : S'T - ST = NX + X'N : X'N - NX,$$

i.e.  $2CT : 2CS = 2CX : 2CN.$

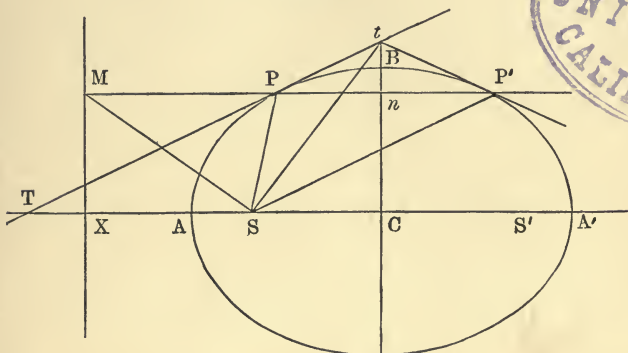
Hence  $CT \cdot CN = CS \cdot CX = CA^2.$

**58. Prop. IV.** *If the tangent at any point  $P$  of an ellipse meet the minor axis  $CB$  produced in  $t$ , and  $Pn$  be the perpendicular on the axis; then will  $Cn \cdot Ct = CB^2$ .*

Let  $Pn$  produced cut the curve again in  $P'$ , and the directrix in  $M$ . Then, since every chord perpendicular to the minor axis is bisected by that axis, it follows, as in Art. 18, Cor. II., that the tangents at  $P, P'$  will meet on the minor axis, and therefore at the point  $t$ .

Then, we know [Art. 10 and Art. 17] that  $SM$  and  $St$  are respectively the external and the internal bisectors of the angle  $PSP'$ , and are therefore at right angles.

Hence  $\angle CSt = \text{complement of } \angle XSM$   
 $= \angle SMX$ .



The right-angled triangles  $CSt$  and  $XMS$  are therefore similar, and we have

$$Ct : CS = SX : XM = SX : Cn;$$

$$\therefore Ct \cdot Cn = CS \cdot SX = BC^2.$$

### EXAMPLES.

1. Shew that, if a focus of an ellipse, the length of its major axis, and one point on the curve be given, the locus of the centre will be a circle.

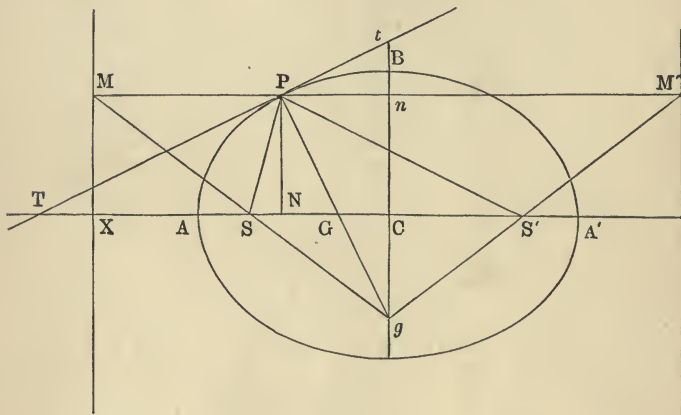
2. One focus of an ellipse and the corresponding directrix are given, and it is also known that a given straight line touches the ellipse. Find the other focus.

3. Find the locus of the centre of an ellipse which has a given focus and which touches a given straight line at a given point.

4. A number of ellipses have a common major axis, and are cut by any line perpendicular to that axis. Shew that the tangents at all the points of section will meet in a point on the major axis.

5. The tangent and the ordinate of any point  $P$  on an ellipse cut the major axis  $CA$  in the points  $T$  and  $N$  respectively. Shew that  $NA$  is less than  $AT$ .

**59. Prop. V.** *If the normal at a point  $P$  of an ellipse cut the major and minor axes respectively in the points  $G, g$ ; then will the ratio  $PG : Pg$  be constant. Also, if  $PN$  and  $Pn$  be the perpendiculars on the axis major and axis minor respectively, then will the ratios  $CG : CN$  and  $Cg : Cn$  be constant.*



Join  $P$  to the foci  $S, S'$  of the ellipse. Draw through  $P$  the line  $MPM'$  perpendicular to and terminated by the directrices. Then, since  $MM'$  and  $SS'$  are both bisected by the minor axis, it follows that  $MS, M'S'$  when produced will meet on the minor axis. Let them meet at the point  $g$ , and let  $Pg$  cut the major axis in the point  $G$ .

Then  $SG : MP = Gg : Pg = GS' : PM'$ ;

$$\therefore SG : GS' = MP : PM'$$

$$= SP : PS'.$$

Hence  $PGg$  bisects the angle  $SPS'$ , and must therefore [Prop. II.] be the *normal* at  $P$ .

Then from similar triangles,

$$PG : Pg = MS : Mg = XS : XC;$$

$$\therefore PG : Pg = CS.SX : CS.CX = CB^2 : CA^2.$$

$$\begin{aligned} \text{Again} \quad CG : CN &= CG : nP = Gg : Pg \\ &= Sg : Mg \\ &= CS : CX. \end{aligned}$$

$$\begin{aligned} \text{Also} \quad Cg : Cn &= gG : GP = gS : SM \\ &= CS : SX. \end{aligned}$$

**60. Prop. VI.** *The circle through the foci of an ellipse and any point  $P$  on the curve passes through the points in which the minor axis is cut by the tangent and normal at  $P$ .*

Let the circle  $SPS'$  cut the minor axis in the points  $t, g$ ; then these points will clearly be on opposite sides of  $SS'$ . Let  $P$  and  $g$  be on opposite sides of  $SS'$ .

Then, since  $tg$  bisects  $SS'$  at right angles, it is the diameter of the circle, and the arcs  $Sg, S'g$  must be equal, whence the angles  $SPg, S'Pg$  are equal. Hence  $Pg$  is the *normal* at  $P$ , and since  $tg$  is the diameter of the circle,  $tPg$  is a right angle, and therefore  $Pt$  is the *tangent* at  $P$  to the ellipse.

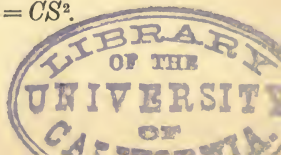
**Cor.** Since  $S, S', g, t$  are on a circle,

$$gC.Ct = S'C.CS = CS^2.$$

Also, since the triangles  $GCg$  and  $tCT$  are similar,

$$CG : gC = Ct : CT.$$

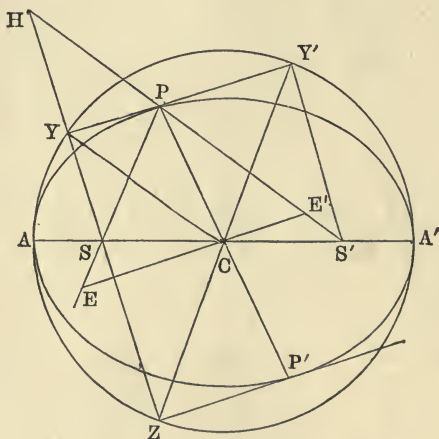
$$\text{Thus} \quad CG.CT = gC.Ct = S'C.CS = CS^2.$$



**61. Prop. VII.** *Shew that the feet of the perpendiculars from the foci on the tangent at any point of an ellipse lie on a fixed circle, and that the semi-minor axis is a mean proportional to the lengths of the perpendiculars.*

Let  $SY, S'Y'$  be the perpendiculars from the foci on the tangent at any point  $P$  of an ellipse.

Join  $SP, S'P$  and let  $S'P, SY$  produced meet in  $H$ . Join  $CY$ .



Then  $\angle SPY = \angle S'PY' = \angle HPY$ .

Also  $\angle SYP = \text{rt. angle} = \angle HYP$ ,

and  $PY$  is common to the two triangles  $SPY, HPY$ .

Hence the two triangles  $SPY, HPY$  are equal in all respects, so that

$$SY = YH \text{ and } SP = PH.$$

Hence  $S'H = S'P + PH = S'P + PS = 2CA$ .

But, since  $SH = 2SY$  and  $SS' = 2SC$ ,

it follows that  $CY$  is parallel to  $S'P$ , and

$$2CY = S'H = 2CA.$$

Thus  $CY = CA$ , and therefore the point  $Y$  is on the circle whose centre is  $C$  and radius  $CA$ .

It can be proved in a similar manner that  $CY'$  is parallel to  $SP$  and equal to  $CA$ .

*Thus the feet of the perpendiculars from the foci on any tangent to an ellipse are on the circle of which the major axis is a diameter.*

**Def.** The circle of which the major axis of an ellipse is a diameter is called the **auxiliary circle**.

Now produce  $Y'C$  to meet the auxiliary circle again in the point  $Z$ . Then, since  $Y'CZ$  is a diameter of the auxiliary circle, the angle  $Y'YZ$  is a right angle. The angle  $Y'YS$  is also a right angle, and therefore  $YSZ$  is a straight line.

Since  $CZ = CY'$ ,  $CS = CS'$  and  $\angle SCZ = \angle S'CY'$ , it follows that the triangles  $SCZ$ ,  $S'CY'$  are equal in all respects, so that  $S'Y' = SZ$ .

Hence  $SY \cdot S'Y' = SY \cdot ZS = AS \cdot SA' = BC^2$ .

**Cor. I.** *If a line through  $C$  parallel to the tangent at  $P$  cut  $PS$ ,  $PS'$ , produced if necessary, in the points  $E$ ,  $E'$  respectively; then  $PE = PE' = AC$ .*

For,  $CE$  is parallel to  $PY'$  and  $CY'$  parallel to  $PSE$ ; hence  $PE = CY' = CA$ . Similarly  $PE' = CY = CA$ .

**Cor. II.** *The rectangle contained by the perpendiculars from a focus on two parallel tangents to an ellipse is constant.*

The converse of Prop. VII. is important, namely that if  $S$  be any point within a given circle and  $S$  be joined to any point  $Y$  on the circle, then the line through  $Y$  perpendicular to  $YS$  will always touch the ellipse of which  $S$  is a focus and of which the given circle is the auxiliary circle.

## EXAMPLES.

1. The portion of the minor axis intercepted between the tangent and the normal at any point of an ellipse can never be less than the distance between the foci.

2. Shew that  $PG$  touches the circle  $SPM$ .

3. Shew that the circles  $PSM$ ,  $PS'M'$  touch one another.

4. Shew that the triangles  $PSg$ ,  $MPg$  are similar, and that the ratio  $Sg : Pg$  is constant.

5. Shew that the triangles  $GSP$ ,  $gS'P$  are similar, and that  
 $PG \cdot Pg = SP \cdot S'P$ .

6. Shew that the triangles  $SPT$ ,  $tPS'$  are similar, and that  
 $SP \cdot S'P = TP \cdot Pt$ .

7. Draw a tangent to an ellipse parallel to a given straight line.

8. Find the foci of an ellipse, having given the tangent at a given point and also the auxiliary circle.

9. Construct an ellipse having given the foci and one tangent.

10. Construct an ellipse having given three tangents and one focus.

11. Shew that the circle on  $SP$  as diameter will touch the auxiliary circle.

12. Any tangent to an ellipse is cut by the tangents at its vertices in the points  $T$ ,  $T'$ . Shew that the circle on  $TT'$  as diameter will pass through the foci.

13. Shew that  $YEE'Y'$  is a parallelogram.

14. Shew that  $SY'$  and  $S'Y$  intersect in the middle point of  $PG$ .

15. Shew that the circle  $YCY'$  passes through the foot of the ordinate of  $P$ .

16. Shew that  $PN$  bisects the angle  $YNY'$ .

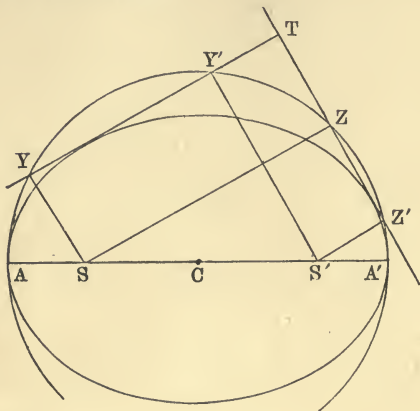
17.  $SY$ ,  $SZ$  are the perpendiculars from a focus on the tangent and normal at any point of an ellipse. Shew that  $YZ$  passes through the centre of the ellipse.

18. An ellipse of given eccentricity has a given focus and touches a given straight line. Shew that its centre is on a fixed circle.

19. Two ellipses have a common focus and equal minor axes; shew that their common tangents are parallel.

20. Two pairs of parallel tangents are drawn to an ellipse and parallels to them are drawn through a focus; shew that the four points where the parallels through the focus meet the tangents lie on a circle.

**62. Prop. VIII.** *The point of intersection of two tangents to an ellipse which are at right angles to one another is on a fixed circle.*



Let  $T$  be the point of intersection of two perpendicular tangents. Draw  $SY$ ,  $S'Y'$  perpendicular to one tangent and  $SZ$ ,  $S'Z'$  perpendicular to the other.

Then  $TZ \cdot TZ' = SY \cdot S'Y' = BC^2$ .

Hence, as  $Z$ ,  $Z'$  are on the auxiliary circle, the square of the tangent from  $T$  to the auxiliary circle is equal to  $BC^2$ .

Hence  $CT^2 - CA^2 = CB^2$ .

Thus  $T$  is on the circumference of the circle the square of whose radius is equal to  $AC^2 + BC^2$ .

**Def.** The circle which is the locus of the point of intersection of perpendicular tangents to an ellipse is called the **Director Circle**.

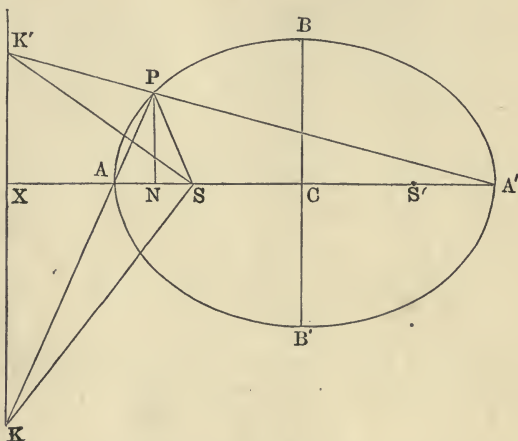
**Ex. 1.** The director-circle of an ellipse cannot cut the directrix in real points.

**Ex. 2.** The length of the tangent to the director-circle of an ellipse drawn from any point on the directrix is equal to the distance of that point from the focus.

**Ex. 3.** Find the locus of the centre of an ellipse whose axes are of given length, and which touches two fixed perpendicular straight lines.

**63. Prop. IX.** *If  $PN$  be the perpendicular from any point  $P$  of an ellipse on the major axis  $AA'$ , then will the ratio  $PN^2 : AN \cdot NA'$  be constant.*

Let  $PA, PA'$  produced cut a directrix in the points  $K, K'$  respectively. Join  $K, K'$  to the corresponding focus  $S$ .



Then  $KS$  bisects the angle  $PSA'$ , and  $K'S$  bisects the angle  $PSA$  [Art. 10]. Hence  $KS$  and  $K'S$  are at right angles, and therefore

$$K'X \cdot XK = XS^2.$$

Now, from similar triangles,

$$PN : NA' = K'X : XA'$$

and

$$PN : AN = XK : XA,$$

$$\therefore PN^2 : AN \cdot NA' = K'X \cdot XK : XA' \cdot XA \\ = XS^2 : XA \cdot XA'.$$

Thus the ratio  $PN^2 : AN \cdot NA'$  is constant for all positions of  $P$ . If  $P$  be taken at an extremity of the minor axis,  $PN$  will be  $BC$  and  $AN \cdot NA'$  will be  $AC^2$ ; thus the constant ratio must be equal to  $BC^2 : AC^2$ .

Hence  $PN^2 : AN \cdot NA' = BC^2 : AC^2.$

**Cor.** If  $Pn$  be the perpendicular from  $P$  on the minor axis, then  $NP = Cn$  and  $Pn = NC$ ;

also  $AN \cdot NA' = CA^2 - CN^2 = CA^2 - Pn^2$ .

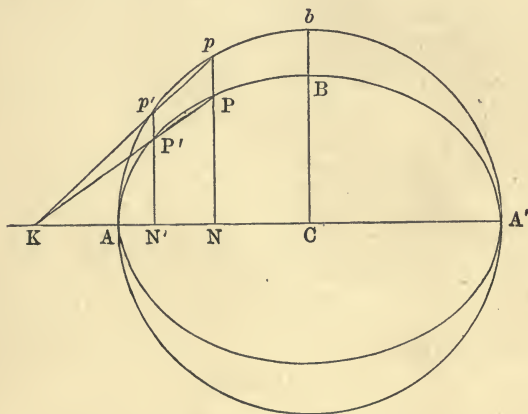
Hence  $Cn^2 : CA^2 - Pn^2 = BC^2 : CA^2$ , 1

or  $Cn^2 : BC^2 = CA^2 - Pn^2 : CA^2$ ;

$\therefore BC^2 - Cn^2 : BC^2 = Pn^2 : CA^2$ .

Hence  $Pn^2 : BC^2 - Cn^2 = CA^2 : BC^2$ .

**64. Prop. X.** If  $PN$  be the ordinate of any point  $P$  on an ellipse and  $NP$  produced cut the auxiliary circle in the point  $p$ ; then  $pN : PN = BC : CA$ .



For, since  $PN^2 : AN \cdot NA' = BC^2 : CA^2$ ,  
and  $pN^2 = Cp^2 - CN^2 = CA^2 - CN^2$   
 $= AN \cdot NA'$ ,

it follows that  $PN^2 : pN^2 = BC^2 : CA^2$ ,  
or  $PN : pN = BC : CA$ .

If the ordinate of an ellipse at a point  $P$  is produced to meet the auxiliary circle in the point  $p$ , the points  $P, p$  are called **corresponding** points.

*The tangents at corresponding points on an ellipse and its auxiliary circle intersect on the major axis.*

For let  $P, P'$  be any two points on an ellipse, and let  $p, p'$  be the two corresponding points on the auxiliary circle. Let  $PP'$  cut the major axis in  $K$ .

$$\begin{aligned}\text{Then} \quad NK : N'K &= NP : N'P' \\ &= Np : N'p',\end{aligned}$$

whence it follows that  $Kp'p$  is a straight line.

Now let  $N'P'p'$  move up to and ultimately coincide with  $NPp$  and it follows that the tangents at  $P, p$  will intersect on the major axis.

The circle on the minor axis as diameter is called the **minor auxiliary circle**. The properties of this circle are not however of much importance.

If the perpendicular  $Pn$  on the minor axis cut the minor auxiliary circle in the point  $Q$ , it follows from Prop. IX. Cor. that

$$Pn : Qn = AC : BC.$$

**65.** If a line be drawn through  $P$  parallel to  $Cp$  and cutting the major and minor axes respectively in the points  $H, K$ . Then  $pCKP$  will be a parallelogram and therefore  $PK = Cp = CA$ . Also  $PH : PN = pC : pN$ , or  $PH : pC = PN : pN = BC : CA$ . Hence  $PH = BC$ . Thus  $PH, PK$  and  $HK$  are all of constant length.

**Conversely**, if a line  $HK$  of constant length have its extremities on two fixed perpendicular lines, any other fixed point  $P$  on the line, or the line produced, will describe an ellipse whose semi-axes are equal to  $KP, HP$  respectively. This is the principle of the *Elliptic Compasses*.

#### EXAMPLES.

1.  $QQ'$  is one of a system of parallel chords of a circle, and  $P$  is a point on  $QQ'$  such that  $QP : PQ'$  is constant. Shew that the locus of  $P$  is an ellipse.
2. Find the foci of an ellipse, having given the extremities of the major axis and one point on the curve.
3. Shew that, if  $NP$  be the ordinate of any point on an ellipse, the locus of the middle point of  $NP$  is another ellipse.
4. Shew that, if  $PP'$  be any chord of an ellipse parallel to either axis, and  $Q$  be the point on  $PP'$  such that  $PQ : QP'$  is a given ratio, the locus of  $Q$  will be another ellipse.
5.  $Q$  is any point on a given circle, and  $QM$  is the perpendicular drawn from  $Q$  on a fixed tangent to the circle. Shew that, if  $P$  be the middle point of  $QM$ , the locus of  $P$  will be an ellipse.

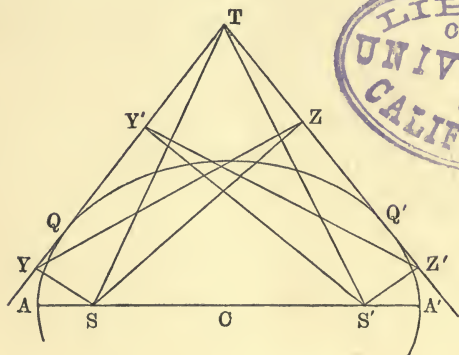
**66. Prop. XI.** *If  $TQ, TQ'$  be tangents to an ellipse whose foci are  $S, S'$ , the angles  $STQ, S'TQ'$  will be equal.*

Draw from the foci  $SY, S'Y'$  perpendicular to  $TQ$ , and  $SZ, S'Z'$  perpendicular to  $TQ'$ ; and join  $Y, Z$  and  $Y', Z'$ .

Then  $\angle YSZ = \angle Y'S'Z'$ , since each is supplementary to  $QTQ'$ .

And, since  $SY \cdot S'Y' = BC^2 = SZ \cdot S'Z'$ ,

$$SY : SZ = S'Z' : S'Y'.$$



Hence the triangles  $YSZ, Z'S'Y'$  are similar, and

$$\angle SZY = \angle S'Y'Z'.$$

But  $S, Y, T, Z$  are cyclic, since  $SYT$  and  $SZT$  are right angles;

$\therefore$  the angles  $STY$  and  $SZY$  are equal or supplementary.

Similarly the angles  $S'TZ'$  and  $S'Y'Z'$  are equal or supplementary.

Hence  $\angle STQ = \angle S'TQ'$ , for they obviously cannot be supplementary.

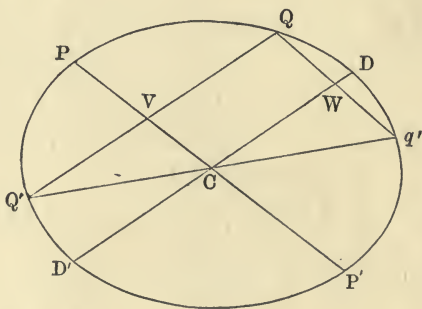
Since  $\angle QTS = \angle Q'TS'$ , it follows that the internal and external bisectors of the angle  $STS'$  are also the internal and external bisectors of the angle  $QTQ'$ .

## PROPERTIES OF DIAMETERS.

**67.** We have already proved that the locus of the middle points of any system of parallel chords of a conic is a straight line through the centre of the conic, which is called a diameter of the conic. We have also proved that the tangents at the extremities of any chord intersect on the diameter which bisects the chord.

Now it is obvious that any line through the centre of an ellipse must cut the curve in two real points; and we have proved in Art. 18, that the tangents at the extremities of any diameter are parallel to the system of chords bisected by that diameter.

**68. Prop. XII.** *If the diameter  $PCP'$  bisect all chords of an ellipse parallel to the diameter  $DCD'$ , then will the diameter  $DCD'$  bisect all chords of the conic which are parallel to  $PCP'$ .*



Let  $QQ'$  be any chord of the ellipse parallel to  $DCD'$ ; then  $V$ , the middle point of  $QQ'$  is on  $PCP'$ .

Join  $Q'C$  and produce it to cut the ellipse again in  $q'$ . Join  $Qq'$  cutting  $DCD'$  in  $W$ .

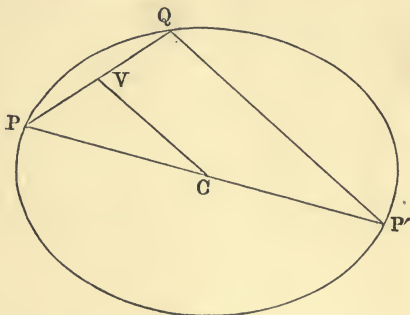
Then, since  $Q'Q$  is bisected in  $V$  and  $Q'q'$  is bisected in  $C$ , it follows that  $Qq'$  is parallel to  $PCP'$ .

Also, since  $CD$  is parallel to  $QQ'$  and bisects  $Q'q'$ ,  $CD$  must also bisect  $Qq'$ .

Thus  $CD$  bisects the chord  $Qq'$  which is parallel to  $PCP'$ , and it must therefore bisect every chord which is parallel to  $PCP'$ .

**Def.** If two diameters of a conic are such that each bisects all chords of the conic which are parallel to the other, the two diameters are said to be **conjugate** to one another.

**69. Prop. XIII.** *The lines joining any point of an ellipse to the two extremities of any diameter are parallel to conjugate diameters.*



Let  $PCP'$  be any diameter of the ellipse, and  $Q$  be any point on the curve.

Join  $QP$ ,  $QP'$  and let  $V$  be the middle point of  $QP$ .

Then, since  $V$  is the middle point of  $PQ$  and  $C$  the middle point of  $PP'$ ,  $CV$  is parallel to  $QP'$ .

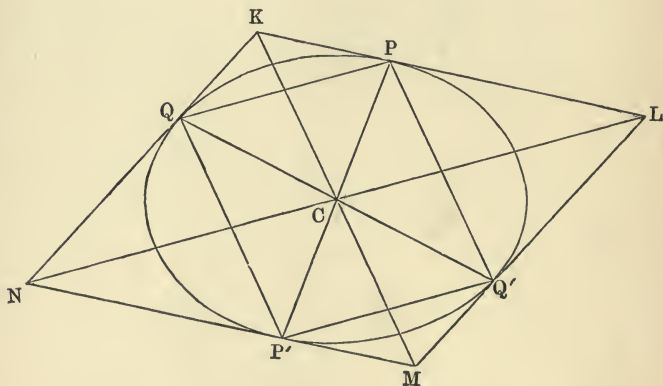
Thus the diameter conjugate to  $PQ$  is parallel to  $QP'$ , as was to be proved.

Conversely, if  $P$ ,  $Q$ ,  $P'$  be three points on an ellipse, such that  $QP$ ,  $QP'$  are parallel to some pair of conjugate diameters, then will  $PP'$  be a diameter.

**Def.** The straight lines joining any point on an ellipse to the two extremities of any diameter, are called **supplemental chords**.

**70. Prop. XIV.** *If the sides of a parallelogram touch an ellipse, its diagonals will be conjugate diameters of the ellipse.*

Let  $KLMN$  be any parallelogram whose sides touch an ellipse, and let  $P, P'$  be the points of contact of one pair of parallel tangents, and  $Q, Q'$  the points of contact of the other pair.



Then, since the tangents at  $P, P'$  are parallel,  $PCP'$  and similarly  $QCQ'$ , is a diameter, and therefore  $PQP'Q'$  is a parallelogram, so that  $PQ$  and  $P'Q'$  are parallel.

Now  $CK$  bisects  $PQ$ , and therefore also the parallel chord  $P'Q'$ . Hence  $KCM$  is a straight line, and it is clearly parallel to  $PQ'$  or  $QP'$ .

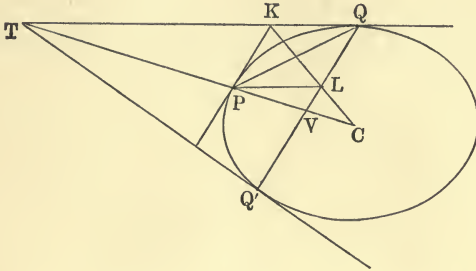
Similarly  $LCN$  is a straight line, and  $CL$  bisects the chord  $PQ'$  which is parallel to  $KCM$ .

Hence  $KM$  and  $LN$  are conjugate diameters of the ellipse.

**Conversely,** *if any two conjugate diameters of an ellipse are cut by any tangent in the points  $K, L$ , the other tangents to the ellipse from  $K$  and  $L$  will be parallel.*

**71. Prop. XV.** *If the tangents at the extremities of a chord  $QQ'$  of an ellipse meet in  $T$ , and the diameter  $CT$  cut  $QQ'$  in  $V$  and the curve in  $P$ ; then will*

$$CV \cdot CT = CP^2.$$



We know [Art. 18] that the tangent at  $P$  is parallel to  $QQ'$ . Let this tangent cut  $TQ$  in  $K$ .

Draw  $PL$  parallel to  $TQ$  meeting  $QQ'$  in  $L$ .

Then, since  $PKQL$  is a parallelogram,  $KL$  will bisect  $PQ$ .

But we know [Art. 18] that  $KC$  will bisect  $PQ$ , whence it follows that  $KLC$  is a straight line.

Then, since  $LV$  is parallel to  $KP$ ,

$$CV : CP = CL : CK.$$

And, since  $PL$  is parallel to  $TQ$ ,

$$CL : CK = CP : CT.$$

Hence  $CV : CP = CP : CT$ ,

or  $CV \cdot CT = CP^2$ .\*

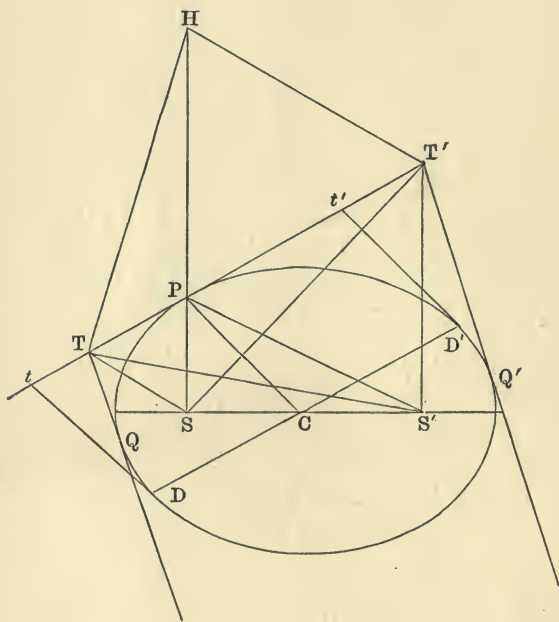
Prop. III. and Prop. IV. are particular cases of the above general theorem.

\* This proof was first given by Dr C. Taylor.

**72. Prop. XVI.** *If the tangent at any point  $P$  of an ellipse whose foci are  $S, S'$  be cut by any pair of parallel tangents in the points  $T, T'$ , and  $CD$  be the semi-diameter conjugate to  $CP$ ; then will*

$$TP \cdot PT' = SP \cdot S'P = CD^2.$$

Let  $Q, Q'$  be the points of contact of the parallel tangents. Join  $ST, S'T, ST', S'T'$ . Produce  $SP$  to  $H$  making  $PH = PS'$ , and join  $HT, HT'$ .



Then, since  $TT'$  bisects the angle  $HPS'$  and  $PH = PS'$ , it follows that  $TH = TS'$  and  $HT' = S'T'$  and therefore that the triangles  $HTT', S'TT'$  are equal in all respects.

$$\begin{aligned}
 \text{Hence} \quad \angle HTT' &= \angle T'TS' \\
 &= \angle STQ; & [\text{Prop. XI.}] \\
 \therefore \angle HTS &= \angle QTT'.
 \end{aligned}$$

$$\text{Similarly} \quad \angle HT'S = \angle Q'T'T.$$

$$\begin{aligned}
 \text{Hence} \quad \angle HTS + \angle HT'S &= \angle QTT' + \angle Q'T'T \\
 &= 2 \text{ right angles, since } TQ \text{ and } T'Q' \text{ are parallel.}
 \end{aligned}$$

Thus  $S, T, H, T'$  lie on a circle, and therefore

$$TP \cdot PT' = SP \cdot PH = SP \cdot S'P.$$

Now let the tangent at  $P$  be cut in  $t, t'$  by the two tangents parallel to  $CP$ ; then

$$\begin{aligned}
 SP \cdot S'P &= tP \cdot Pt' \\
 &= CD^2, \text{ since } tP = Pt' = CD.
 \end{aligned}$$

$$\text{Thus} \quad TP \cdot PT' = SP \cdot S'P = CD^2.$$

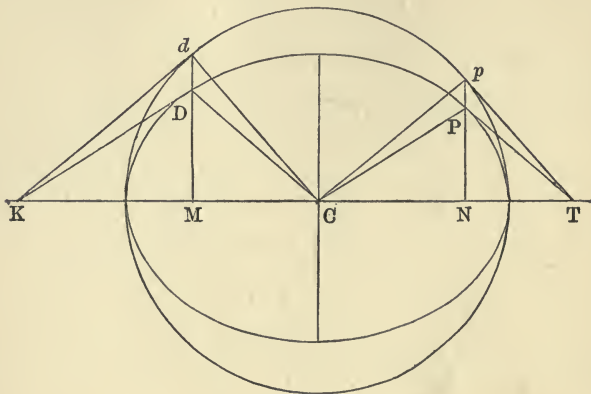
**Cor.** Since the diagonals of a parallelogram whose sides touch an ellipse are conjugate diameters, the proposition may be enunciated thus:

*If the tangent at  $P$  to an ellipse be cut in  $T, T'$  by any pair of conjugate diameters; then will*

$$TP \cdot PT' = SP \cdot S'P = CD^2.$$

It should be noticed that since the angles  $TST'$  and  $THT'$  are supplementary, the angles  $TST'$  and  $TS'T'$  are supplementary; and therefore *the portion of any tangent to an ellipse intercepted between any pair of parallel tangents, or between any pair of conjugate diameters, subtends supplementary angles at the foci.*

**73. Prop. XVII.** *The sum of the squares of conjugate diameters of an ellipse is constant.*



Let  $CP, CD$  be any pair of conjugate semi-diameters of an ellipse. Draw the tangents at  $P, D$  cutting the axis major in  $T, K$  respectively; and let  $PN, DM$  be perpendicular to the axis major.

Then  $PT$  is parallel to  $CD$  and  $DK$  parallel to  $CP$ .

Produce  $NP, MD$  to meet the auxiliary circle in  $p$  and  $d$  respectively.

Then, since  $CN \cdot CT = CA^2 = Cp^2$ , it follows that  $pT$  must be the tangent at  $p$  to the auxiliary circle. Similarly  $Kd$  touches the auxiliary circle at  $d$ .

Since  $PT$  is parallel to  $CD$ , the right-angled triangles  $TNP, CMD$  must be similar.

$$\begin{aligned} \text{Hence} \quad TN : CM &= NP : MD \\ &= Np : Md, \end{aligned}$$

whence it follows that the triangles  $TNp$  and  $CMd$  are similar, and therefore that  $Cd$  is parallel to  $Tp$ . But  $Tp$  touches the auxiliary circle and is therefore perpendicular to  $Cp$ .

Hence  $Cp$  and  $Cd$  are at right angles, whence it follows that  $CM = Np$  and  $CN = Md$ .

$$\text{Now} \quad CP^2 + CD^2 = CN^2 + NP^2 + CM^2 + MD^2.$$

$$\text{But} \quad CN^2 + CM^2 = CN^2 + Np^2 = CA^2;$$

$$\text{and} \quad NP^2 + MD^2 : Np^2 + Md^2 = BC^2 : CA^2;$$

$$\text{hence, as} \quad Np^2 + Md^2 = Np^2 + CN^2 = CA^2,$$

$$NP^2 + MD^2 = CB^2.$$

$$\text{Thus} \quad CP^2 + CD^2 = CA^2 + CB^2.$$

**Or thus**

$$(SP + S'P)^2 = SP^2 + S'P^2 + 2SP \cdot S'P.$$

$$\text{But} \quad SP + S'P = 2CA,$$

$$SP \cdot S'P = CD^2, \quad [\text{Prop. XVI.}]$$

$$\text{and} \quad SP^2 + S'P^2 = 2CP^2 + 2CS^2.$$

$$\text{Hence} \quad 4CA^2 = 2CP^2 + 2CS^2 + 2CD^2;$$

$$\begin{aligned} \therefore CP^2 + CD^2 &= 2CA^2 - CS^2 = CA^2 + (CA^2 - CS^2) \\ &= CA^2 + CB^2. \end{aligned}$$

### EXAMPLES.

1. Shew that two diameters of an ellipse which make equal angles with either axis are equal to one another.

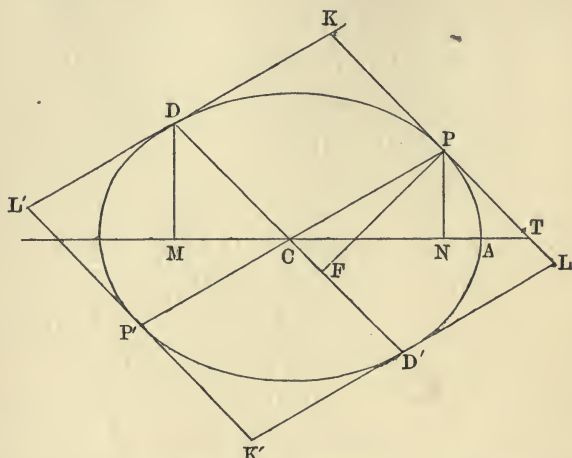
2. Shew that the equal conjugate diameters of an ellipse are parallel to the lines joining an extremity of one axis to the two extremities of the other.

3. Shew that if a parallelogram be inscribed in an ellipse, the diagonals of the parallelogram will be diameters of the ellipse.

4. Shew that, if  $CP$ ,  $CD$  be conjugate semi-diameters of an ellipse, and if the tangents at  $P$ ,  $D$  meet in  $T$ , and  $CT$  cut the curve in  $R$  and  $PD$  in  $V$ , then will  $2CV^2 = CR^2$  and  $CT^2 = 2CR^2$ . Hence find the loci of  $V$  and  $T$  for different pairs of conjugate diameters.

5. Shew that, if  $P$ ,  $D$  be any two points on an ellipse, and  $p$ ,  $d$  the corresponding points on the auxiliary circle, and if the tangents at  $P$ ,  $D$  meet in  $T$  and the tangents at  $p$ ,  $d$  meet in  $t$ ; then will  $Tt$  be perpendicular to the major axis of the ellipse. Hence shew that, if  $pCd$  be a constant angle, the locus of  $T$  will be an ellipse.

**74. Prop. XVIII.** *The parallelogram formed by the tangents at the ends of conjugate diameters of an ellipse, is of constant area.*



Let the tangents at the extremities of the conjugate diameters  $PCP'$ ,  $DCD'$  form the parallelogram  $KK'L'L'$ . Let the tangent at  $P$  cut the major axis in  $T$ , and draw  $PN$ ,  $DM$  perpendicular to the axis.

$$\begin{aligned}\text{Then} \quad \parallel^m KK' &= 4 \parallel^m KC = 8 \triangle DCP \\ &= 8 \triangle DCT \\ &= 4DM \cdot CT.\end{aligned}$$

Now it can be proved as in Prop. XVII. that

$$DM : CN = BC : AC;$$

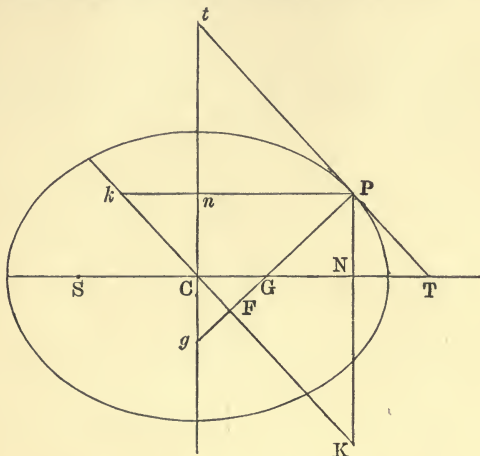
$$\therefore DM \cdot CT : CN \cdot CT = BC \cdot AC : AC^2.$$

But  $CN \cdot CT = CA^2$  and therefore  $DM \cdot CT = BC \cdot AC$ .

Thus the parallelogram formed by the tangents at the ends of any pair of conjugate diameters of the ellipse is constant and equal to  $4AC \cdot BC$ .

**Cor.** *If the normal at  $P$  cut the conjugate diameter in the point  $F$ , then  $PF \cdot CD = AC \cdot BC$ .*

**75. Prop. XIX.** *If the normal at any point  $P$  of an ellipse cut the major and minor axes respectively in the points  $G, g$ , and the diameter conjugate to  $CP$  in the point  $F$ ; then  $PF \cdot PG = BC^2$  and  $PF \cdot Pg = AC^2$ .*



Let the tangent at  $P$  cut the major and minor axes respectively in the points  $T, t$ .

Draw  $PN, Pn$  perpendicular to the axes, and produce to meet the conjugate diameter in the points  $K, k$  respectively.

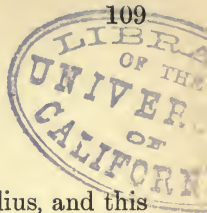
Then a circle will circumscribe  $GFKN$ , since the angles at  $N$  and  $F$  are right angles;

$$\begin{aligned}\therefore PF \cdot PG &= PN \cdot PK \\ &= Cn \cdot Ct, \text{ since } NP = Cn \text{ and } KP = Ct \\ &= BC^2.\end{aligned}$$

Again, a circle will circumscribe  $gFnk$ , since the angles  $gFk$  and  $gnk$  are right angles;

$$\begin{aligned}\therefore PF \cdot Pg &= Pn \cdot Pk \\ &= CN \cdot CT, \text{ since } CN = nP \text{ and } CT = kP \\ &= CA^2.\end{aligned}$$





Hence  $2PF \cdot PO = CP^2 + CD^2$ ,  
from which  $PO$  can be found.

Then, having found  $PO$  from the relation

$$2PF \cdot PO = CP^2 + CD^2,$$

draw a circle with  $O$  as centre and  $OC$  as radius, and this circle will cut the normal at  $P$  in the points  $G, g$  on the axes. Thus the directions of the axes are found, and the lengths of the semi-axes are given by the relations

$$PF \cdot PG = BC^2,$$

and  $PF \cdot Pg = AC^2$ .

Having found the axes of the ellipse, the foci and directrices can be easily found.

### EXAMPLES.

1. Shew that the area of a parallelogram inscribed in an ellipse is greatest when its diagonals are conjugate diameters.

2. Shew that the area of a parallelogram whose sides touch an ellipse cannot be less than that of the parallelogram formed by the tangents at the ends of the axes.

**78.** We have already proved in Art. 24 that the ratio of the rectangles contained by the segments of any two intersecting chords of an ellipse, which are parallel to two given straight lines respectively, is constant for all positions of the point of intersection of the chords; and, by considering the parallel chords through the centre of the ellipse, it follows that *the ratio of the rectangles contained by the segments of any two chords of an ellipse is equal to the ratio of the squares of the parallel semi-diameters*; and, as a particular case, *the lengths of the two tangents drawn from any point to an ellipse are in the ratio of the parallel semi-diameters*.

**79. Prop. XX.** *The length of a focal chord of an ellipse varies as the square of the parallel semi-diameter.*

Let  $PSP'$  be the focal chord and  $DCD'$  the parallel diameter.



Then, since the line joining the middle points of any two parallel chords of a circle is perpendicular to the chords, it follows that the chords  $KL$  and  $MN$  are perpendicular to the diameter of the ellipse which is conjugate to them, and therefore the chords must be parallel to one of the axes.

**Conversely**, if two chords of an ellipse, which are not parallel, be equally inclined to an axis, their four extremities will lie on a circle.

## EXAMPLES.

1. If  $OP$ ,  $OQ$  be two tangents to an ellipse whose semi-axes are  $CA$ ,  $CB$ ,  $OP : OQ$  cannot be greater than  $CA : CB$ .

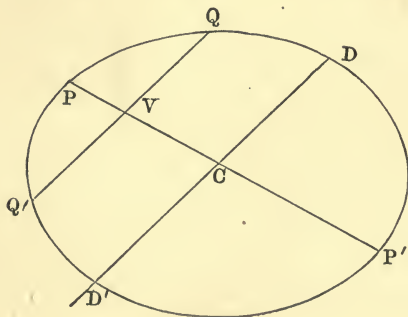
2. Shew that a circle cannot cut an ellipse in more than four points.

3. If the chords  $PQ$ ,  $PQ'$  of an ellipse be equally inclined to an axis, the circle  $PQQ'$  will touch the ellipse at  $P$ .

**81. Def.** A line  $QV$ , drawn from any point  $Q$  of an ellipse parallel to the tangent at either extremity of the diameter  $PVCP'$ , is called **an ordinate to the diameter  $PCP'$** .

**Prop. XXII.** *If  $QV$  be the ordinate to any diameter  $PVCP'$  of an ellipse, and  $CD$  be the semi-diameter conjugate to  $PCP'$ ; then will*

$$QV^2 : CP^2 - CV^2 = CD^2 : CP^2.$$



Let  $QV$  produced cut the ellipse in  $Q'$ ; then, since  $QQ'$  is parallel to the tangent at  $P$ ,  $V$  will be the middle point of  $QQ'$ .

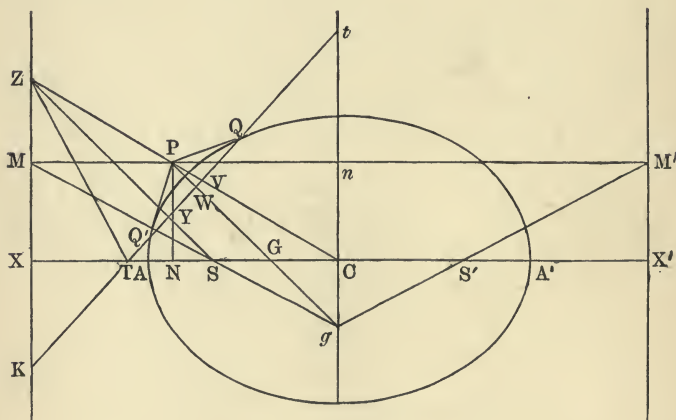
Now we know [Art. 78] that the rectangles contained by the segments of two chords of an ellipse drawn in given directions is constant. Hence

$$QV \cdot VQ' : PV \cdot VP' = DC \cdot CD' : PC \cdot CP';$$

$$\therefore QV^2 : PV \cdot VP' = DC^2 : PC^2,$$

or 
$$QV^2 : CP^2 - CV^2 = CD^2 : CP^2.$$

**82.** The following generalizations of some of the preceding theorems are interesting.



Let  $QQ'$  be any chord of an ellipse. Draw  $SY$  perpendicular to the chord and let  $SY$  produced cut the directrix in  $Z$ .

Then we know that  $CZ$  will bisect the chord and that the tangents at  $Q$ ,  $Q'$  will meet at some point  $P$  on  $CZ$ .

Draw the perpendiculars  $PM$ ,  $PN$  on the directrix and transverse axis respectively. Draw through  $P$  a line perpendicular to  $QQ'$  meeting  $QQ'$  in  $W$ , the major axis in  $G$  and the minor axis in  $g$ . Let  $QQ'$  cut the axes in  $T$ ,  $t$ .

Then, since  $PG$  is parallel to  $SZ$ ,

$$CG : CS = CP : CZ$$

$$= CN : CX, \text{ since } PN \text{ and } ZX \text{ are parallel.}$$

Hence  $\mathbf{CG : CN = CS : CX} \dots\dots\dots(\text{i}).$

Also  $SG : SC = ZP : ZC$

$$= PM : CX;$$

$$\therefore \mathbf{SG : PM = SC : CX} \dots\dots\dots(\text{ii}).$$

Again, since  $SG : PM = SC : CX$

$$= S'G : M'P,$$

it follows that  $PG$  goes through the intersection of  $MS$  and  $M'S'$ ;

$$\therefore Cn : gC = XM : gC;$$

$$\therefore \mathbf{Cn : gC = XS : SC} \dots\dots\dots(\text{iii}).$$

Also, since

$$PG : Pg = MS : Mg;$$

$$\therefore \mathbf{PG : Pg = XS : XC = BC^2 : CA^2} \dots\dots\dots(\text{iv}).$$

Now, since  $TK$  is perpendicular to  $SZ$ , and  $ZK$  perpendicular to  $SZ$ , it follows that  $SK$  is perpendicular to  $ZT$ .

But we know that  $SK$  is perpendicular to  $SP$ .

Hence  $\mathbf{ZT \text{ is parallel to } SP} \dots\dots\dots(\text{v}).$

Then  $CN : CX = CP : CZ$

$$= CS : CT, \text{ since } ZT \text{ is parallel to } SP;$$

$$\therefore \mathbf{CN . CT = CS . CX = CA^2} \dots\dots\dots(\text{vi}).$$

Also

$$CG : CS = CP : CZ$$

$$= CS : CT, \text{ from (v);}$$

$$\therefore \mathbf{CG . CT = CS^2} \dots\dots\dots(\text{vii}).$$

Again, since  $PGg$  is perpendicular to  $Tt$ , and  $TG$  perpendicular to  $tg$ , it follows that  $G$  is the orthocentre of the triangle  $Ttg$ , and therefore that

$$\mathbf{tC . Cg = CG . CT = CS^2} \dots\dots\dots(\text{viii}).$$

And, since  $tC . Cg = SC . CS'$ , it follows that  $S, t, S', g$  are on a circle; and  $tg$  must be the diameter of this circle, since  $tg$  bisects  $SS'$  at right angles, whence it follows that  $W$  is also on the circle.

Thus  $\mathbf{S, S', t, g \text{ and } W \text{ are a circle}} \dots\dots\dots(\text{ix}).$

From (iii) and (vii) it follows that

$$\mathbf{Cn . Ct = CB^2} \dots\dots\dots(\text{x}).$$

Moreover, if a line through  $C$  parallel to the chord  $OQ'$  cut  $PGg$  in  $F$ , it can be easily proved, as in Art. 75, that

$$\mathbf{PF . Pg = BC^2, \text{ and } PF . Pg = AC^2} \dots\dots\dots(\text{xi}).$$

When  $P$  is on the ellipse the relations (i), (ii), (iii), &c. reduce to those proved in Articles 57, 58, 59, &c.

## EXAMPLES ON THE ELLIPSE.

1. Tangents are drawn to an ellipse from any point on the line through a focus perpendicular to the axis. Shew that the length intercepted by the tangents on the corresponding directrix is bisected by the axis.

2. A semi-circular piece of paper is folded over so that a particular point  $K$  of the bounding diameter lies somewhere on the circular boundary: prove that the line of the crease touches an ellipse whose foci are  $K$  and  $C$ ,  $C$  being the centre of the circle.

3. From any point on the auxiliary circle of an ellipse tangents are drawn touching the ellipse in  $P$ ,  $Q$ , and  $PCP'$ ,  $Q C Q'$  are diameters of the ellipse. Shew that  $PQ'$  and  $P'Q$  are focal chords.

4. Two tangents  $TP$ ,  $TQ$  are drawn from any point  $T$  to an ellipse, and any straight line parallel to  $TP$  meets  $TQ$  in  $L$ ,  $PQ$  in  $M$ , and the conic in  $R$ ,  $R'$ . Shew that  $LM^2 = LR \cdot LR'$ .

5. The tangent at a point  $P$  of an ellipse whose foci are  $S$ ,  $S'$  cuts the tangents at the vertices in the points  $T$ ,  $T'$ , and  $TS$ ,  $T'S'$  meet in  $Q$ . Shew that  $Q$  is on the normal at  $P$  and that the circle  $SPS'$  is the nine point circle of the triangle  $QTT'$ .

6. From the foci  $S$ ,  $S'$  of an ellipse lines are drawn perpendicular to  $SP$ ,  $S'P$  respectively, and meeting the normal at  $P$  in  $O$ ,  $O'$  respectively. Shew that  $OO'$  is bisected by the minor-axis.

7.  $PSQ$ ,  $P'S'Q'$  are parallel focal chords of an ellipse; shew that the tangents at  $P$ ,  $Q$ ,  $P'$ ,  $Q'$  form a parallelogram two of whose angular points are on the directrix and the remaining two on the auxiliary circle.

8. Shew that, if  $P$ ,  $D$  be extremities of a pair of conjugate diameters of an ellipse, the lines joining  $P$  and  $D$  to the foci will touch a circle whose radius is equal to the semi-minor axis.

9.  $P$ ,  $p$  are corresponding points on an ellipse and its auxiliary circle. Shew that the perpendicular distances of the foci  $S$ ,  $S'$  from the tangent to the circle at  $p$  are equal to  $SP$ ,  $S'P$  respectively.

10. The sides of the parallelogram  $ABCD$  touch an ellipse whose focus is  $S$ ; shew that the circles  $ABS$ ,  $BCS$ ,  $CDS$ ,  $DAS$  are all equal.

11. If a focal chord of a conic pass through the extremities of a pair of conjugate diameters of an ellipse, the length of the chord will be equal to the semi-axis-major.

12. A line  $AB$  is drawn from a fixed point  $A$  to meet a fixed circle in  $B$ ; through  $B$  a line  $BC$  is drawn perpendicular to  $AB$  to meet a concentric circle in  $C$ . Shew that a line through  $C$  parallel to  $AB$  will touch a fixed conic.

13. Any line through the focus of an ellipse cuts the tangents at the vertices in the points  $T, T'$  respectively. Shew that the circle whose diameter is  $TT'$  touches the ellipse at the ends of a chord parallel to the major axis.

14. If two ellipses have the same auxiliary circle and one pass through the foci of the other, the second will pass through the foci of the first.

15.  $AA'$  is the major axis of an ellipse of which  $S, S'$  are the foci and  $P$  any point on the curve;  $AR, A'R'$  are drawn parallel to  $SP, S'P$  respectively to meet the tangent at  $P$  in  $R, R'$ . Shew that

$$AR + A'R' = AA'.$$

16. A parabola is described passing through the foci of a given ellipse and having for focus any point on the ellipse; shew that the directrix of the parabola always touches the auxiliary circle of the ellipse.

17. The tangents at  $P, D$  the extremities of conjugate diameters of an ellipse whose centre is  $C$  meet in  $T$ , and  $CT$  cuts the ellipse in  $Q$ , and the chords  $QR, QR'$  are drawn parallel to  $CP, CD$  respectively. Shew that  $RR'$  is parallel to  $PD$ .

18. Lengths  $CA, CB$  are taken on any two given straight lines so that the sum of the squares of  $CA$  and  $CB$  is equal to a given square, and the parallelogram  $ACBP$  is completed. Shew that the locus of  $P$  is an ellipse.

19.  $CY$  is the perpendicular from the centre on the tangent at any point  $P$  of an ellipse and  $Q$  is the point of contact of the other tangent from  $Y$ . Shew that the normal at  $P$  passes through the other extremity of the diameter through  $Q$ .

20. Perpendiculars  $SY, S'Y'$  are drawn from the foci  $S, S'$  upon a pair of tangents  $TY, T'Y'$  to an ellipse; prove that the angles  $STY, S'TY'$  are equal or complementary to the angles at the base of the triangle  $CYY'$ .

21. If the ordinate at a point  $P$  of an ellipse meet the auxiliary circle in  $p$ , shew that the diameter of the circle through  $p$  and the normal at  $P$  to the ellipse meet on a fixed circle.

22. Any point  $P$  of an ellipse is joined to  $A, A'$ , the extremities of the major axis, and  $AF$  is drawn perpendicular to  $A'P$ .  $AP$  and  $AF$  meet the tangent at  $A'$  in  $K$  and  $L$ . Prove that  $A'K : A'L = BC^2 : AC^2$ .

23. Through any point  $P$  of an ellipse a line is drawn perpendicular to the diameter  $CP$  meeting the auxiliary circle in  $R, R'$ . Shew that  $RR'$  is equal to the differences of the focal distances of an extremity of the diameter conjugate to  $CP$ .

24. Through a point  $P$  of an ellipse a line  $PDE$  is drawn cutting the axes so that the segments  $PD$  and  $PE$  are equal to the two semi-axes respectively: perpendiculars to the axes through  $E$  and  $D$  intersect in  $O$ . Prove that  $O$  is a normal.

25.  $P$  is any point on an ellipse whose centre is  $C$  and whose foci are  $S, S'$ . The diameter conjugate to  $CP$  cuts  $SP$  in  $E$ . Shew that the difference of the squares of  $CP$  and  $SE$  is constant.

26. An ellipse of given semi-axes touches three of the sides of a given rectangle; find its centre and focus.

27. If a chord of an ellipse be parallel to one of the equi-conjugate diameters, the normals at its extremities will meet on the diameter of the ellipse which is perpendicular to the other equi-conjugate diameter.

28. If  $CP, CD$  are conjugate semi-diameters of an ellipse, and the normals at  $P, D$  meet in  $O$ ; then will  $CO$  be perpendicular to  $PD$ .

29. The normals at  $P, P'$ , the extremities of a focal chord of an ellipse, cut the axis major in  $G, G'$  respectively and intersect in the point  $O$ . Shew that the line through  $O$  parallel to  $PP'$  will bisect  $GG'$ .

30. Two given ellipses in the same plane have a common focus and one revolves about the common focus while the other remains fixed: prove that the locus of the point of intersection of their common tangents is a circle.

31. A focus  $S$  of an ellipse, the length of the major axis, and one point  $P$  on the curve are given. Shew that the ellipse always touches a fixed ellipse whose foci are  $S$  and  $P$ .

32. Having given the focus of an ellipse, the length of the major axis, and that the second focus lies on a fixed straight line, prove that the ellipse will touch two fixed parabolas having the given focus for focus.

33. The centre of an ellipse, the radius of its director-circle, and one point on the curve are given. Shew that the ellipse always touches a fixed concentric ellipse.

34. Two given concentric ellipses in the same plane have equal major axes. Shew how to draw their common tangents.

35. Shew that, if two of the diagonals of a quadrilateral circumscribing an ellipse intersect in a focus, these diagonals must be at right angles and the third diagonal must be the corresponding directrix.

36. If  $PP'$  be a chord of an ellipse parallel to the major axis, and the two circles be drawn through a focus  $S$  touching the conic at  $P, P'$  respectively; prove that the circles will intersect at a point  $F$  which is the intersection of  $PP'$  and  $ST$ , where  $T$  is the point of intersection of the tangents at  $P, P'$ .

Prove also that the locus of  $F$ , for different positions of  $PP'$ , will be a parabola whose vertex is  $S$ .

37. A circle cuts off equal chords  $AB, CD$  from two given parallel straight lines, and passes through a fixed point  $S$  between the lines; prove that the intersecting straight lines  $AD, BC$  always touch a fixed conic of which  $S$  is one focus.

38. If  $P, p$  be corresponding points on an ellipse and the auxiliary circle, centre  $C$ , and if  $CP$  be produced to meet the auxiliary circle in  $q$ , prove that the tangent at the point  $Q$  on the ellipse corresponding to  $q$  is perpendicular to  $Cp$ , and that it cuts off from  $Cp$  a length equal to  $CP$ .

39. Two ellipses lie in the same plane and have one common focus and equal major axes; one ellipse revolves in its own plane about the common focus, the other ellipse remaining fixed. Prove that the common chord of the two ellipses always touches another ellipse confocal with the fixed ellipse.

40. The tangent at any point  $P$  of an ellipse meets any pair of parallel tangents in  $M, N$ . Prove that the circle described on  $MN$  as diameter will meet the normal at  $P$  at points whose distance apart is equal to the diameter conjugate to  $CP$ , and whose distances from the centre are respectively equal to the sum and the difference of the semi-axes.

41. The diameter parallel to any focal chord of an ellipse is equal to the chord joining the points on the auxiliary circle corresponding to the extremities of the focal chord.

42. The sides of a rectangle touch an ellipse; shew that a circle through a focus and any two adjacent angular points of the rectangle is equal to the auxiliary circle.

43. Any tangent to an ellipse cuts the director-circle in the points  $P, Q$  and a directrix in the point  $K$ . Shew that, if  $S$  be the corresponding focus, the triangles  $KSP, SKQ$  will be similar.

44. The area of the parallelogram formed by the tangents at the extremities of any two diameters of an ellipse varies inversely as the area of the parallelogram formed by joining the points of contact.

45. A system of parallelograms is inscribed in an ellipse whose sides are parallel to the equi-conjugate diameters: prove that the sum of the squares of the sides of any parallelogram is constant.

46. An ellipse is described having the line joining the foci of a given ellipse for a diameter and for that conjugate to it a line equal to the minor axis of the given ellipse: prove that it always touches the original ellipse.

47. A series of ellipses have their foci on two adjacent sides of a given parallelogram and touch the other two sides. Shew that the locus of their centres is a straight line.

48. An extremity of a diameter of a central conic is joined to the two extremities of one of its ordinates: prove that the chords so drawn are proportional to the diameters parallel to them.

49. Normal chords of a conic which are at right angles are proportional to the parallel diameters.

50. If  $PP'$  be a chord of an ellipse normal at  $P$ ,  $CY$  the perpendicular on the tangent at  $P$ , and  $CQ$  the semi-diameter parallel to  $PP'$ ; then will  $PP' \cdot CY = 2CQ^2$ .

51.  $PCP'$  is any diameter of an ellipse,  $PQ$  is any chord which is produced so as to cut the tangent at  $P'$  in  $R$ . Shew that the diameter parallel to  $PQ$  is a mean proportional between  $PQ$  and  $PR$ .

52. From any point on an ellipse two chords are drawn equally inclined to the tangent at the point. Shew that these chords are to one another as the parallel focal chords.

53. An ellipse of given major axis has one focus at the focus of a given parabola and touches the parabola; shew that the locus of the centre of the ellipse is a straight line perpendicular to the axis of the parabola. Shew also that all the ellipses touch another parabola co-axial and confocal with the given one.

54. The tangent at any point  $P$  of an ellipse cuts the directrices corresponding to the foci  $S, S'$  in the points  $Z, Z'$  respectively. Shew that  $ZS, Z'S'$  will meet on the ordinate through the other extremity of the diameter through  $P$ .

55. The line joining the foci of an ellipse subtends at the pole of any chord an angle equal to half the sum of the angles which it subtends at the extremities of the chord.

56. The ordinate at any point  $P$  of an ellipse is produced to meet the perpendicular from the centre on the tangent at  $P$  in  $Q$ . Shew that the locus of  $Q$  is an ellipse.

57.  $PP'$  is a diameter of an ellipse and  $Q$  is a point such that the tangents at  $P$  and  $Q$  are at right angles. Shew that  $PQ, P'Q$  make equal angles with the tangent at  $Q$ . Prove also that if  $R$  be any other point on the ellipse  $QP + QP'$  is greater than  $RP + RP'$ .

58. Shew that the perimeter of a parallelogram inscribed in an ellipse cannot be greater than that of the parallelogram whose diagonals are the axes of the ellipse.

59.  $PP'$  is any chord of an ellipse and  $PG, P'G'$  are the normals at  $P, P'$ , the points  $G, G'$  being on the major-axis. Shew that the projections of  $PG, P'G'$  on the chord  $PP'$  are equal.

60. Construct an ellipse having given the centre, a tangent, the length of the major-axis, and a point through which a directrix passes.

61. The chord of contact of the tangents drawn from a point  $O$  to an ellipse cuts the major axis in  $T$ , and the line through  $O$  perpendicular to the chord cuts the major axis in  $G$ . Shew that the circle whose diameter is  $TG$  is cut orthogonally by a fixed circle.

62. Shew that, if any point  $P$  on an ellipse be joined to the foci and the joining lines be produced to meet the corresponding directrices in  $R, R'$ ; then  $RR'$  and the tangent at the other extremity of the diameter through  $P$  will meet on the axis of the ellipse.

63.  $SY, S'Y'$  are the perpendiculars from the foci on the tangent at any point  $P$  of an ellipse;  $X, X'$  are the feet of the directrices corresponding to the foci  $S, S'$  respectively. Shew that  $XY, X'Y'$  will meet on the minor axis, and that  $NY, NY'$  will be perpendicular to  $XY, X'Y'$ ,  $N$  being the foot of the ordinate through  $P$ .

64. If  $AA'$  is the transverse axis of an ellipse, and if  $Y, Y'$  are the feet of the perpendiculars let fall from the foci on the tangent at any point of the curve, prove that the locus of the point of intersection of  $AY$  and  $A'Y'$  is an ellipse.

65. Any two conjugate diameters of an ellipse cut the director circle in the points  $R, R'$ ; shew that  $RR'$  touches the ellipse.

66. Shew that, if  $SY, S'Y'$  be the perpendiculars from the foci on the tangent at any point  $P$  of an ellipse, the tangents to the auxiliary circle at  $Y, Y'$  will meet in a point  $T$  on the ordinate  $NP$ , and the locus of  $T$  will be an ellipse.

67.  $SY, S'Y'$  are the perpendiculars from the foci  $S, S'$  on any tangent to an ellipse, and  $X, X'$  are the feet of the corresponding directrices;  $XY, X'Y'$  cut the auxiliary circle again in  $Z, Z'$  respectively. Shew that  $ZZ'$  will touch the ellipse.

68. The area of a triangle whose vertices are any three points on a given ellipse is in a constant ratio to the area of the triangle whose vertices are the three corresponding points on the auxiliary circle.

69. Shew that if a triangle inscribed in an ellipse be of maximum area, its centroid will coincide with the centre of the ellipse.

70. Having given one focus of an ellipse, the directrix corresponding to the other focus, and also a tangent to the ellipse. Find the centre.

71. A system of ellipses is drawn having a common focus  $S$  and a common latus-rectum  $LSL'$ . A fixed straight line through  $S$  is drawn to intersect the conics and the normals are drawn at the points of intersection. Shew that these normals all touch a parabola whose focus lies on  $LSL'$ .

72. The minor-axis of an ellipse inscribed in a given triangle cannot exceed the diameter of the inscribed circle.

73. An ellipse touches the sides of a given triangle and has a given centre. Find the points of contact.

74. If  $PSQ$  be a focal chord of an ellipse, and  $Pp, Qq$  be chords perpendicular to it, then the triangles  $SPp, SQq$  will be similar.

75. Shew that the exterior angle between two tangents to an ellipse is equal to half the sum of the angles the chord of contact subtends at the foci.

76. If the double ordinate  $PP'$  drawn perpendicular to the major axis of an ellipse whose centre is  $C$  meet the auxiliary circle in  $p$  and  $p'$ , shew that the part of the normal at  $P$  intercepted between  $Cp$  and  $Cp'$  is bisected at  $P$ .

77. The normal to an ellipse at  $P$  cuts the axes in  $G, g$ , and  $CK$  is the central perpendicular on the tangent at  $P$ . Shew that, if  $O, O'$  be the middle points of  $CG, Cg$  respectively, then  $OB=OK=OP$  and  $O'A=O'K=O'P$ .

78.  $A$  is a fixed point within a given circle,  $P$  is any point on the circumference of the circle and  $PQ$  is drawn making a given angle with  $AP$ . Shew that  $PQ$  touches a fixed ellipse, and that the eccentricity of the ellipse is independent of the value of the fixed angle  $APQ$ .

79.  $PNP'$  is any double ordinate of an ellipse whose centre is  $C$ , and the normal at  $P$  cuts  $CP'$  in  $Q$ . Shew that the locus of  $Q$ , for different positions of  $PNP'$ , is an ellipse.

80. If a pair of tangents to an ellipse be at right angles to one another, the rectangle contained by the perpendiculars from the centre and the intersection of the tangents on the chord of contact is constant.

81. A parallelogram circumscribes a given circle, and two of the angular points are on fixed straight lines parallel to one another and equidistant from the centre; shew that the other two are on an ellipse of which the given circle is the minor auxiliary circle.

82. From the centre  $C$  of two concentric circles two radii  $CQ, Cq$  are drawn equally inclined to a fixed straight line, the first to the outer circle, the second to the inner: prove that the locus of the middle point  $P$  of  $Qq$  is an ellipse, that  $PQ$  is the normal at  $P$  to this ellipse, and that  $Qq$  is equal to the diameter conjugate to  $CP$ .

83. From  $P$  any point on an ellipse a tangent is drawn to the minor auxiliary circle cutting the director circle in  $Q, R$ ; shew that  $PQ, QR$  are equal to the focal distances of  $P$ .

84. From a point  $T$  the tangents  $TP, TQ$  are drawn to an ellipse. Shew that, if the bisector of the angle  $PTQ$  passes through a fixed point on the major axis of the ellipse,  $T$  must lie on a fixed circle.

85. Shew that, if a circle roll on the inside of a circle of double the radius, any point carried with the moving circle will describe an ellipse.

86. Shew that, if  $P$  be any point on an ellipse whose foci are  $S, S'$ , the locus of the centre of the circle inscribed in the triangle  $SPS'$  is an ellipse.

87. The portion of any normal chord of an ellipse intercepted between the directrices subtends at the pole of the chord an angle equal to half the sum of the angles subtended at the extremities of the chord by the distance between the foci.

88.  $TQ, TQ'$  are any two tangents to an ellipse, and  $TN$  is the perpendicular on the axis major. Shew that  $TN$  bisects the angle  $QNQ'$ .

89. The tangent at a fixed point  $P$  of an ellipse whose foci are  $S, S'$  is cut by any pair of parallel tangents in the points  $T, T'$ . The lines  $TS, T'S'$  intersect in  $O$ , and the lines  $TS', T'S$  in  $O'$ ; shew that  $O, O'$  lie on a fixed circle through the foci.

90. Three conics are inscribed in an acute-angled triangle, and the three points of bisection of the perpendiculars from the angles on the sides of the triangle are each a focus of one conic; prove that these three points form with the other three foci a hexagon of which the opposite sides meet in the angles of the triangle.

91.  $PG$  is the normal at any point  $P$  of an ellipse and  $GL$  is drawn perpendicular to  $CP$ , and  $CM$  is drawn parallel to one of the focal distances of  $P$  meeting  $PG$  in  $M$ . Shew that the triangles  $CLM$  and  $MCP$  are similar.

92. Given a tangent to an ellipse, its point of contact, and the director-circle, construct the ellipse.

93. Shew that, if two chords  $QQ'$ ,  $RR'$  of an ellipse be drawn parallel to one of the equi-conjugate diameters, meeting the other equi-conjugate diameter  $PCP'$  in the points  $V$ ,  $W$  on opposite sides of the centre  $C$  such that  $2WC \cdot CV = CP^2$ , then will the normals at the points  $Q$ ,  $Q'$ ,  $R$ ,  $R'$  meet in a point on the diameter perpendicular to  $PCP'$ .

94. An ellipse is inscribed in a triangle and having its centre at the centre of the circum-circle of the triangle. Prove that the perpendiculars from the corners of the triangle on the opposite sides will be normals to the ellipse.

95. With the orthocentre of a triangle as centre two conics are described, one touching the sides and the other passing through the angular points; shew that these conics are similar, their corresponding axes being perpendicular.

96.  $PSQ$ ,  $PHR$  are focal chords of an ellipse; prove that the tangent at  $P$  and the chord  $QR$  cut the major axis in points equidistant from the centre.

97.  $PSQ$ ,  $PHR$  are focal chords of an ellipse, and the tangents at  $Q$ ,  $R$  meet in  $T$ . Shew that the middle point of  $TP$  is on the minor axis and that the locus of  $T$  is an ellipse.

98. Two ellipses whose major axes are perpendicular intersect in four points; shew that these four points lie on a circle.

99. Two ellipses whose major axes are parallel intersect in four points; shew that the four points will lie on a circle, unless the ellipses have the same eccentricity.

100. If  $OP$ ,  $OQ$  are two tangents to an ellipse and  $CP'$ ,  $CQ'$  be the parallel semi-diameters, shew that  $OP \cdot OQ + CP' \cdot CQ' = OS \cdot OH$ .

## CHAPTER IV.

### THE HYPERBOLA.

83. FROM the definition of a hyperbola as the locus of a point which moves in the plane containing a given point, called the focus, and a given straight line, called the directrix, in such a manner that its distance from the focus is always in a constant ratio greater than unity to its perpendicular distance from the directrix, we have already proved in Chapter I. that a hyperbola is symmetrical about the line through its focus perpendicular to its directrix; and also that, if the line cut the curve in the points  $A$ ,  $A'$ , and  $C$  be the middle point of  $AA'$ , the hyperbola is symmetrical about the line  $BCB'$  drawn through  $C$  parallel to the directrix, whence it follows that there is a second focus in the line  $SAA'$  and a second directrix perpendicular to  $AA'$ .

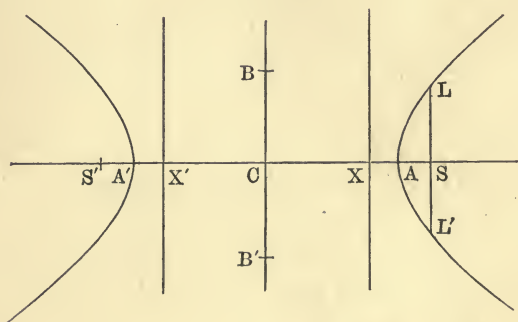
We have also proved in Art. 4 that, if  $S$ ,  $S'$  be the two foci, and  $SAA'S'$  cut the directrices in the points  $X$ ,  $X'$  respectively, then

$$CS : CA = CA : CX = SA : AX.$$

The curve consists of two separate branches, as in the figure, no part of the curve lying between the lines drawn through the vertices parallel to the directrices. [Art. 6.]

The lines  $ACA'$  and  $BCB'$  about each of which the hyperbola is symmetrical are called respectively its **transverse** and **conjugate** axes.

The axis  $ACA'$  cuts the curve in two points and the intercepted portion  $AA'$  is also called *the transverse axis*.



The line  $BCB'$  does not cut the curve in real points; but if points  $B, B'$  be taken on  $BCB'$  such that  $B'C = CB$

$$\text{and} \quad -BC^2 = CA^2 - CS^2;$$

$$\text{then} \quad BC^2 = CS^2 - CA^2 = (CS + CA)(CS - CA) = A'S.AS.$$

$$\text{Also} \quad BC^2 = CS^2 - CA^2 = CS^2 - CS.CX = CS.XS.$$

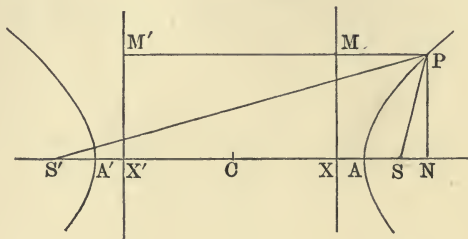
If  $LSL'$  be the latus-rectum, we have

$$\begin{aligned} SL : XS &= SA : AX \\ &= CS : CA. \end{aligned}$$

$$\text{Hence} \quad \mathbf{SL.CA = CS.XS = BC^2.}$$

**84. Prop. I.** *The difference of the focal distances of any point on a hyperbola is constant.*

Let  $S, S'$  be the foci and  $P$  any point on the hyperbola. Join  $SP, S'P$  and draw  $PMM'$  perpendicular to and meeting the directrices in the points  $M, M'$ .



Then

$$SP : MP = CA : CX,$$

and

$$S'P : M'P = CA : CX$$

$$\therefore S'P - SP : M'P - MP = CA : CX.$$

But

$$M'P - MP = M'M = X'X = 2CX.$$

Hence

$$S'P - SP = 2CA.$$

If  $P$  be any point on the branch through  $A$ ,

$$S'P - SP = 2CA;$$

and, if  $P$  be any point on the branch through  $A'$ ,

$$SP - S'P = 2CA.$$

It can easily be proved as in Art. 55 that, if  $Q$  be any point external to the hyperbola, that is if  $Q$  be such that  $SQ$  is cut by the curve in one and only one point between  $S$  and  $Q$ , then

$$S'Q \sim SQ < AA'.$$

Similarly, if  $Q$  be an internal point, that is if  $Q$  be such that  $SQ$  is cut by the curve in two points or in none between  $S$  and  $Q$ , then

$$SQ \sim S'Q > AA'.$$

The above property of a hyperbola enables us to describe the curve by the continuous motion of a point. For, if the end  $A$  of a rod  $AL$  be made to turn about the fixed point  $A$ , and if one end of a string of constant length be fastened to the rod at  $L$  and the other end be fastened to a fixed point  $B$ , and if the string be kept tight by a



pencil moving along the rod  $LA$ ; then, if  $P$  be any position of the point of the pencil,  $LP + PB$  will be equal to the length of the string, and  $LP + PA$  will be equal to the length of the rod, so that  $AP - BP$  will be equal to the constant difference between the length of the string and the length of the rod; and therefore  $P$  will always be on a hyperbola whose foci are  $A$  and  $B$ .

### EXAMPLES.

1. Construct a conic having given a focus and the two extremities of its transverse axis.

2. Find the locus of the centre of a circle (1) which touches a given line and a given circle, (2) which passes through a given point and touches a given circle, and (3) which touches two given circles.

3. Find the locus of the centre of a circle which cuts two given circles so that the common chords may each be equal to a given straight line.

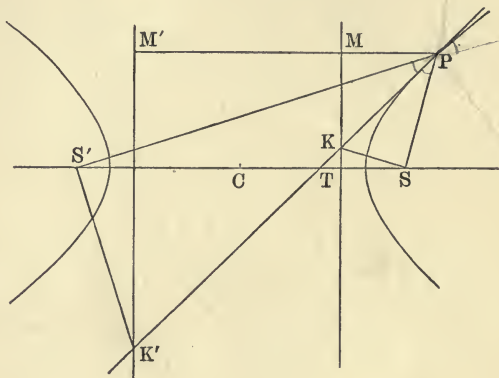
4. On a plane field the crack of the rifle and the thud of the ball striking the target are heard at the same instant: find the locus of the hearer.

5. The centre of a hyperbola, the length of the transverse axis, and one point on the curve are given. Shew that the locus of the foci is another hyperbola.

**85. Prop. II.** *The tangent at any point of a hyperbola is equally inclined to the focal distances of the point.*

Let the tangent at  $P$  cut the directrices corresponding to  $S, S'$  in  $K, K'$ , respectively.

Draw  $PMM'$  perpendicular to the directrices. Join  $SP, S'P, SK, S'K'$ .



Then, from the similar triangles  $MPK, M'PK'$ ,

$$\begin{aligned} KP : K'P &= MP : M'P \\ &= SP : S'P. \end{aligned}$$

Also we know [Art. 12] that the angles  $KSP, K'S'P$  are right angles.

Hence the triangles  $KPS, K'PS'$  are similar, and we have

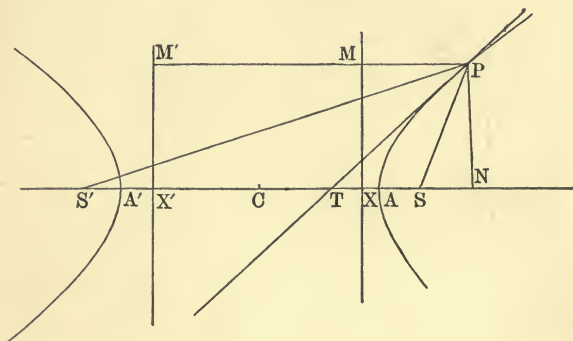
$$\angle SPK = \angle K'PS'.$$

Thus *the tangent at  $P$  bisects the angle  $SPS'$ .*

Since the normal is perpendicular to the tangent, it follows that *the normal bisects the angle between  $SP$  and  $S'P$  produced.*

If the tangent at  $P$  cut the transverse axis in  $T$ ,  $PT$  bisects the angle  $SPS'$  and therefore  $S'T : TS = S'P : PS$ . Now if  $P$  be any point on the branch through the vertex  $A$ ,  $S'P$  will be greater than  $PS$ , and therefore  $S'T$  will be greater than  $TS$ .  $T$  must therefore lie between  $C$  and  $A$ .

**86. Prop. III.** *If the tangent at any point  $P$  of a hyperbola cut the transverse axis in  $T$ , and  $PN$  be the perpendicular on the axis; then will  $CN \cdot CT = CA^2$ .*



Draw  $PMM'$  perpendicular to the directrices. Then, since  $PT$  bisects the angle  $SPS'$ ,

$$S'T : TS = S'P : SP$$

$$= M'P : MP = X'N : XN.$$

Hence  $S'T + TS : S'T - TS = X'N + XN : X'N - XN$ ,

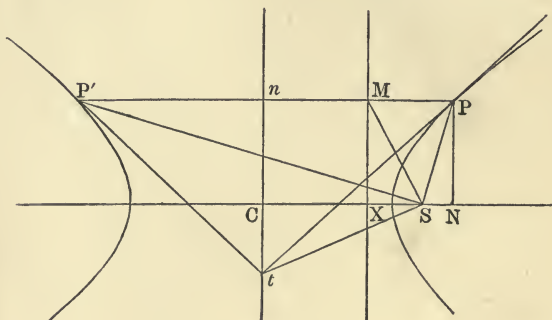
i.e.  $2CS : 2CT = 2CN : 2CX.$

Hence  $CT \cdot CN = CS \cdot CX = CA^2$



**87. Prop. IV.** *If the tangent at any point  $P$  of a hyperbola cut the conjugate axis in  $t$  and  $Pn$  be the perpendicular on the axis; then will the rectangle  $Cn.Ct$  be constant.*

Let  $Pn$  produced cut the curve again in  $P'$ , and the directrix in  $M$ .



Then, since every chord perpendicular to the conjugate axis is bisected by that axis, it follows, as in Art. 18, Cor. II., that the tangents at  $P$ ,  $P'$  will meet on the conjugate axis, and therefore at the point  $t$ .

Then, we know [Art. 10 and Art. 17] that  $SM$  and  $St$  are respectively the internal and external bisectors of the angle  $PSP'$ , and are therefore at right angles.

Hence  $\angle CSt = \text{compt. of } \angle XSM = \angle XMS$ .

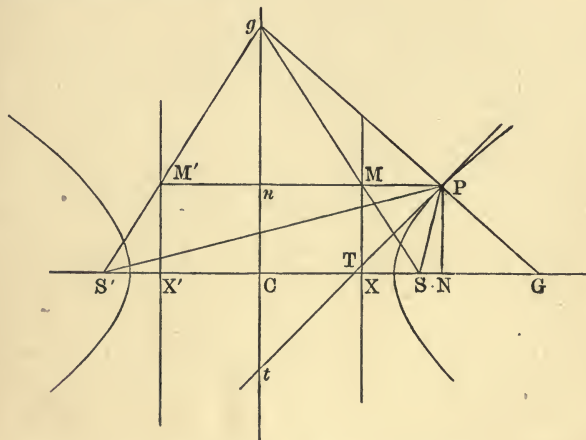
The right-angled triangles  $CSt$  and  $XMS$  are therefore similar, and we have

$$tC : CS = XS : XM = XS : Cn;$$

$$\therefore tC.Cn = CS.XS = BC^2.$$

**88. Prop. V.** *If the normal at a point  $P$  of a hyperbola cut the transverse and conjugate axes respectively in the points  $G, g$ ; then will the ratio  $PG : Pg$  be constant. Also, if  $PN$  and  $Pn$  be the perpendiculars on the transverse and conjugate axes respectively, then will the ratios  $CG : CN$  and  $Cg : Cn$  be constant.*

Join  $P$  to the foci  $S, S'$  of the hyperbola. Draw through  $P$  the line  $PMM'$  perpendicular to and terminated by the directrices. Then, since  $MM'$  and  $SS'$  are both bisected by the conjugate axis, it follows that  $MS, M'S'$  when produced will meet on the conjugate axis. Let them meet in the point  $g$ , and let  $Pg$  cut the transverse axis in the point  $G$ .



$$\begin{aligned} \text{Then } SG : MP &= Gg : Pg = GS' : PM'; \\ \therefore SG : S'G &= PM : PM' \\ &= PS : PS'. \end{aligned}$$

Hence  $PGg$  bisects the angle between  $SP$  and  $S'P$  produced, and must therefore [Prop. II.] be the *normal* at  $P$ .

Then from similar triangles,

$$\begin{aligned} GP : Pg &= SM : Mg = SX : XC; \\ \therefore GP : Pg &= CS . XS : CS . CX \\ &= CB^2 : CA^2. \end{aligned}$$

$$\begin{aligned} \text{Again } CG : CN &= CG : nP = Gg : Pg \\ &= Sg : Mg = CS : CX. \end{aligned}$$

$$\begin{aligned} \text{Also } Cg : Cn &= Gg : GP = Sg : SM \\ &= SC : SX. \end{aligned}$$

**89. Prop. VI.** *The circle through the foci of a hyperbola and any point  $P$  on the curve passes through the points in which the conjugate axis is cut by the tangent and normal at  $P$ .*

Let the circle  $SPS'$  cut the conjugate axis in the points  $t, g$ ; then these points will clearly be on opposite sides of  $SS'$ . Let  $P$  and  $g$  be on the same side of  $SS'$ .

Then, since  $tg$  bisects  $SS'$  at right angles, it is the diameter of the circle, and the arcs  $S't, tS$  must be equal, whence the angles  $S'Pt, tPS$  are equal. Hence  $Pt$  is the *tangent* at  $P$ , and since  $tg$  is the diameter of the circle,  $tPg$  is a right angle, and therefore  $Pg$  is the *normal* at  $P$  to the hyperbola.

**Cor.** Since  $S, S', g, t$  are on a circle,

$$gC . Ct = S'C . CS = CS^2.$$

Also, since the triangles  $GCg$  and  $TCt$  are similar,

$$CG : Cg = Ct : CT.$$

$$\text{Thus } CG . CT = gC . Ct = S'C . CS = CS^2.$$

**90. Prop. VII.** *If the normal at any point  $P$  of a hyperbola cut the transverse and conjugate axes in the points  $G, g$  respectively and the diameter parallel to the tangent at  $P$  in the point  $F$ ; then  $PF \cdot PG = BC^2$  and  $PF \cdot Pg = AC^2$ .*

The proof of Art. 75 is applicable.



### EXAMPLES.

1. Given the focus of a hyperbola, the length of its transverse axis and one point on the curve; find the locus of the other focus and of the centre.

2. One focus of a hyperbola and the corresponding directrix are given, and it is also known that a given straight line touches the curve. Find the other focus.

3. Find the locus of the centre of a hyperbola which has a given focus and touches a given straight line at a given point.

4. One focus of an ellipse is given and also two points on the curve; find the locus of the other focus and also the locus of the centre.

5. Shew that the triangles  $PSg$  and  $MPg$  are similar, and that the ratio  $Pg : Sg$  is constant.

6. Shew that the triangles  $SPg$  and  $GPS'$  are similar, and that

$$PG \cdot Pg = SP \cdot S'P.$$

7. Shew that the triangles  $PST$  and  $PtS'$  are similar, and that

$$PT \cdot Pt = SP \cdot S'P.$$

8. Shew that the angles  $PSt$  and  $PTS$  are supplementary.

9. Find a point  $P$  on a hyperbola such that the circle  $SPS'$  is least.

10. Shew that  $CT \cdot NG = BC^2$  and  $tC \cdot ng = AC^2$ .

11. If the normal at  $P$  cut the conjugate axis in  $g$ , the projection of  $Pg$  on either focal distance will be equal to half the transverse axis.

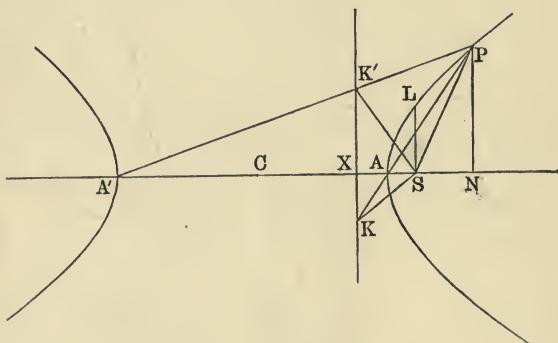
12. Any tangent to a hyperbola cuts the tangents at the extremities of its transverse axis in the points  $T, T'$ . Shew that the circle whose diameter is  $TT'$  passes through the foci.

**91. Prop. VIII.** *If  $PN$  be the perpendicular from any point  $P$  of a hyperbola on the transverse axis  $AA'$ , then will the ratio  $PN^2 : AN \cdot A'N$  be constant.*

Let  $PA, PA'$  cut a directrix in the points  $K, K'$  respectively. Join  $K, K'$  to the corresponding focus  $S$ .

Then  $KS$  bisects the angle between  $SP$  and  $AS$  produced, and  $K'S$  bisects the angle  $A'SP$  [Art. 10]. Hence  $KS$  and  $K'S$  are at right angles, and therefore

$$KX \cdot XK' = XS^2.$$



Now, from similar triangles,

$$PN : AN = KX : AX,$$

and

$$PN : A'N = XK' : XA';$$

$$\begin{aligned} \therefore PN^2 : AN \cdot A'N &= KX \cdot XK' : AX \cdot XA' \\ &= XS^2 : AX \cdot XA'. \end{aligned}$$

Thus the ratio  $PN^2 : AN \cdot A'N$ , that is

$$PN^2 : CN^2 - CA^2,$$

is constant for all positions of  $P$ .

Let  $LS$  be the semi-latus rectum, then the constant ratio is equal to  $LS^2 : CS^2 - CA^2$ .

$$\begin{aligned}\text{Hence } PN^2 : CN^2 - CA^2 &= LS^2 : CS^2 - CA^2 \\ &= LS^2 : BC^2 \\ &= BC^2 : CA^2. \quad [\text{Art. 83.}]\end{aligned}$$

**Conversely**, if  $N$  be any point in the straight line  $AA'$  produced, and  $NP$  be drawn perpendicular to  $AA'$ , and such that the ratio  $NP^2 : AN \cdot A'N$  is constant, the locus of  $P$  will be a hyperbola of which  $AA'$  is the transverse axis.

Ex. 1.  $N$  is any point exterior to a given circle and on the fixed diameter  $AA'$ .  $NP$  is drawn perpendicular to  $AA'$  and equal to the tangent from  $N$  to the circle. Shew that the locus of  $P$  is a hyperbola.

Ex. 2.  $PP'$  is any chord of a given circle perpendicular to the fixed diameter  $AA'$ ; shew the locus of the point of intersection of  $AP$  and  $A'P'$  is a hyperbola.

92. If the relation

$$PN^2 : CN^2 - CA^2 = BC^2 : CA^2,$$

be supposed to hold good for all positions of the ordinate  $PN$ , and if  $Cb$  be the length of the semi-diameter perpendicular to  $ACA'$ , we shall have

$$Cb^2 : -CA^2 = BC^2 : CA^2,$$

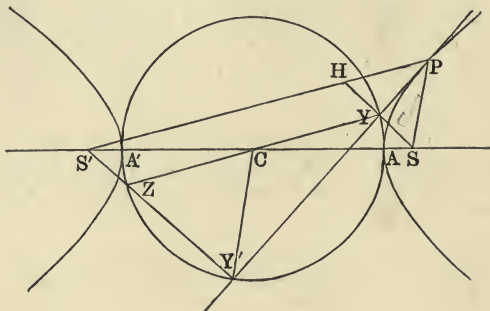
whence it follows that  $Cb^2 = -BC^2$ , or  $Cb = \sqrt{-1} \cdot CB$ .

Thus the imaginary length of the semi-axis conjugate is given by the relation  $Cb^2 = -CB^2 = CA^2 - CS^2$ ; and it should be noticed that the relation between the squares of the lengths of the two axes and the distance between the foci is precisely the same in the hyperbola as in the ellipse.

**93. Prop. IX.** *Shew that the feet of the perpendiculars from the foci on the tangent at any point of a hyperbola lie on a fixed circle, and that the rectangle contained by these perpendiculars is constant.*

Let  $SY, S'Y'$  be the perpendiculars from the foci on the tangent at any point  $P$  of a hyperbola.

Join  $SP, S'P$  and let  $SY$  be produced to meet  $S'P$  in  $H$ . Join  $CY$ .



Then, since  $PY$  bisects the angle  $SPH$ , and  $SYH$  is perpendicular to  $PY$ , it follows that  $SY = YH$  and  $SP = PH$ .

Hence  $S'H = S'P - HP = S'P - SP = 2CA$ .

But, since  $SC = CS'$  and  $SY = YH$ , it follows that  $CY$  is parallel to  $S'HP$ , and  $CY = \frac{1}{2}S'H = CA$ .

Thus the point  $Y$  is on the circle whose centre is  $C$  and radius  $CA$ .

It can be proved in a similar manner that  $CY'$  is parallel to  $SP$  and equal to  $CA$ .

Thus the feet of the perpendiculars from the foci on any tangent to a hyperbola are on the circle of which the transverse axis is a diameter, and which is called the auxiliary circle.

Produce  $YC$  to meet the auxiliary circle in  $Z$ . Then, since  $YCZ$  is a diameter of the auxiliary circle,  $YY'Z$  is a right angle. The angle  $YY'S'$  is also a right angle, and therefore  $S'ZY'$  is a straight line.

Since  $CY = CZ$ ,  $SC = CS'$  and  $\angle SCY = \angle S'CZ$ , it follows that the triangles  $SCY$ ,  $S'CZ$  are equal in all respects, so that  $S'Z = SY$ .

Hence  $YS \cdot S'Y' = S'Z \cdot S'Y' = S'A \cdot S'A'$ .

**Cor. I.** *If a line through  $C$  parallel to the tangent at  $P$  cut  $SP$ ,  $S'P$ , produced if necessary, in the points  $E$ ,  $E'$  respectively; then  $PE = PE' = AC$ .*

For,  $CE$  is parallel to  $PY$  and  $CY$  parallel to  $S'EP$ ; hence

$$PE = CY = CA.$$

**Cor. II.** *The rectangle contained by the perpendiculars from a focus on two parallel tangents to a hyperbola is constant.*

The converse of Prop. IX. is important, namely, that if  $S$  be any point without a given circle whose centre is  $C$  and  $S$  be joined to any point  $Y$  on the circle, then the line through  $Y$  perpendicular to  $YS$  will touch the hyperbola of which  $S$  is a focus and of which the given circle is the auxiliary circle, the point of contact of the tangent being its point of intersection with a line through the other focus parallel to  $CY$ .

#### EXAMPLES.

1. The locus of the centre of a conic which has a given focus and touches two given straight lines is a straight line.

2. One focus of a conic is given and three tangents; find the other focus.

3. Shew that the circumcircle of the triangle formed by three tangents to a conic cannot pass through a focus unless the conic is a parabola.

4. Shew that, if a chord of a given circle subtend a right angle at a given point, the chord will touch a fixed conic of which the given point and the centre of the given circle are the foci.

94. If  $Y$  be the point of contact of either tangent from the focus  $S$  to the auxiliary circle, the line through  $Y$  perpendicular to  $SY$  will, by the converse of the last proposition, be a tangent to the hyperbola; and the line through  $Y$  perpendicular to  $YS$  will pass through the centre of the conic; moreover, the point of contact of this tangent will be its point of intersection with a line through the other focus parallel to  $CY$ , that is parallel to the tangent itself.

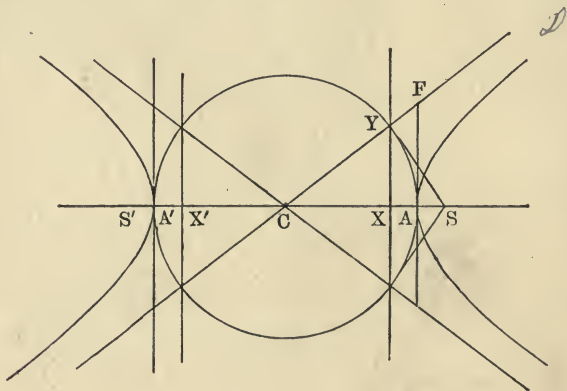
Thus *there are two tangents to a hyperbola which pass through the centre of the curve, and the points of contact of these tangents are at an infinite distance from the centre.*

The above result also follows from Art. 86.

**Def.** A tangent to any curve whose point of contact is at an infinite distance, is called an **asymptote**.

If  $X$  be the foot of the directrix corresponding to the focus  $S$ ,  $CX : CY = CY : CS$ , whence it follows that the triangles  $XCY$ ,  $YCS$  are similar, so that  $CXY$  is a right angle.

Hence *the asymptotes of a hyperbola pass through the points of intersection of the auxiliary circle and either directrix.*



If the tangent at the vertex  $A$  cut an asymptote in  $F$ , the similar triangles  $FAC$ ,  $SYC$  will be equal in all respects, since  $CY = CA$ .

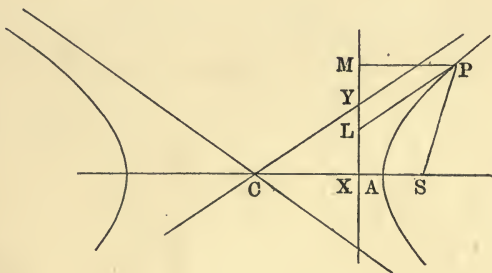
Thus  $CF = CS$ , and therefore

$$SY^2 = AF^2 = CF^2 - CA^2 = CS^2 - CA^2 = CB^2.$$

The angle between the asymptotes will be greater or less than a right angle according as  $BC$  is greater or less than  $CA$ .

When  $BC = CA$  the asymptotes are at right angles to one another, and the hyperbola is called a **rectangular hyperbola**.

**95.** It should be noticed that the focal distances of any point on a hyperbola are equal to its distances from the directrices *measured parallel to an asymptote*.



For, if  $P$  be any point on the curve, and if  $PM$  be perpendicular to the directrix  $XLM$  and  $PL$  be parallel to an asymptote, the triangle  $LPM$  will be similar to the triangle  $YCX$ .

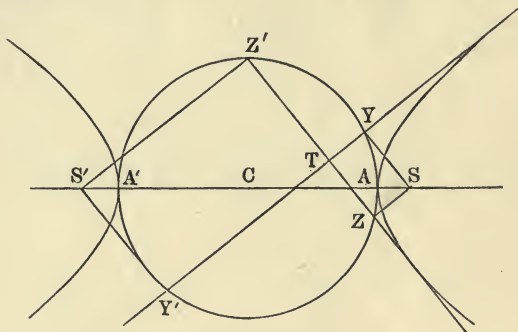
$$\text{Hence } LP : PM = CY : CX = CA : CX = SP : PM;$$

$$\therefore LP = SP.$$

**96. Prop. X.** *The point of intersection of two tangents to a hyperbola which are at right angles to one another is on a fixed circle.*

Let  $Y, Y'$  be the perpendiculars from the foci  $S, S'$  on any tangent; then  $Y, Y'$  will be on the auxiliary circle.

Hence, if a tangent perpendicular to  $YY'$  cut  $YY'$  in  $T$ , this tangent will be parallel to  $SY$  or  $S'Y'$ ; and, since the foci are on *opposite* sides of any tangent to a hyperbola,  $T$  must be between  $Y$  and  $Y'$  and therefore within the auxiliary circle.



Then, if  $SZ, S'Z'$  be the focal perpendiculars on the other tangent through  $T$ ,

$$SZ = YZ \text{ and } S'Z' = Y'T.$$

Hence  $YT \cdot TY' = SZ \cdot Z'S' = BC^2.$

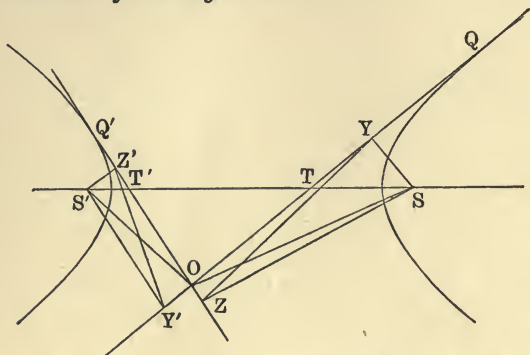
But  $Y, Y'$  are on the auxiliary circle,  
therefore  $BC^2 = YT \cdot TY' = CA^2 - CT^2.$

Thus  $T$  is on the circle the square of whose radius is  $CA^2 - CB^2$ . This circle is called the **Director Circle**.

In the case of a rectangular hyperbola the radius of the director-circle is zero, so that the asymptotes are the only perpendicular tangents.

The director-circle is imaginary when  $CB^2$  is greater than  $CA^2$ , that is when the angle between the asymptotes is greater than a right angle, and in this case no tangent is at right angles to any other.

**97. Prop. XI.** *The tangents drawn from any point  $O$  to a hyperbola whose foci are  $S, S'$ , are equally inclined to the bisectors of the angle  $SOS'$ .*



Draw the focal perpendiculars  $SY, S'Y'$  to  $OQ$  and  $SZ, S'Z'$  to  $OQ'$ . Join  $Y, Z$  and  $Y', Z'$ .

Then  $SY, S'Y'$  are parallel and in opposite directions, and  $SZ, S'Z'$  are also parallel and in opposite directions; whence it follows that the angles  $YSZ$  and  $Y'S'Z'$  are equal.

$$\text{But } SY \cdot S'Y' = BC^2 = SZ \cdot S'Z';$$

$$\therefore SY : SZ = S'Z' : S'Y'.$$

Hence the triangles  $YSZ, Z'S'Y'$  are similar, and

$$\angle SZY = \angle S'Y'Z'.$$

Now  $S, Y, Z, O$  are cyclic, since  $SYO$  and  $SZO$  are right angles; and therefore the angles  $SZY$  and  $SOY$  are either equal or supplementary.

Similarly  $S'Y'Z'$  and  $S'OZ'$  are either equal or supplementary.

Hence  $SOY$  and  $S'OZ'$  are equal or supplementary; and similarly the angles  $SOZ, S'OY'$  are equal or supplementary.

If the tangents cut the transverse axis in the points  $T, T'$  respectively, the points  $T, T'$  will both be between  $S$  and  $S'$ , and therefore the angles  $SOT, S'OT'$  must be equal. Hence the internal and external bisectors of the angle  $SOS'$  are also the internal and external bisectors of the angle  $TOT'$ .

## PROPERTIES OF DIAMETERS.

98. We have already proved that the locus of the middle points of any system of parallel chords of a conic is a straight line through the centre of the conic, which is called a diameter of the conic. We have also proved that the tangents at the extremities of any chord intersect on the diameter which bisects the chord, and that the tangents at the extremities of any diameter are parallel to the system of chords bisected by that diameter.

There is, however, a very important difference between an ellipse and a hyperbola; for in an ellipse every diameter must cut the curve in two real points, but this is by no means true of every diameter of a hyperbola.

If the tangent at any point  $P$  of a hyperbola cut a directrix in  $K$ , we know that  $PK$  subtends a right angle at the corresponding focus  $S$ , and therefore  $SP$  must be less than  $PK$ .

Now if the diameter parallel to the tangent at  $P$  were to cut the curve in  $Q$ ; then, since no part of the curve is between the directrices,  $CQ$  would cut the directrix in some point  $L$  between  $C$  and  $Q$ . Also, since  $C$  is the centre of the curve,  $QC$  would cut the curve in another point  $Q'$  such that  $QC = CQ'$ .

Then, since  $QC$  is parallel to the tangent at  $P$ , it follows from the definition of the curve that we should have

$$SP : PK = SQ : QL = SQ' : LQ' = SQ + SQ' : QL + LQ'.$$

But  $SP < PK$ , and therefore  $SQ + SQ' < QL + LQ'$ , which is impossible since  $L$  is between  $Q$  and  $Q'$ .

Hence *the diameter parallel to the tangent at any point  $P$  of a hyperbola cannot meet the curve in real points.*

*Thus of every pair of conjugate diameters of a hyperbola only one will cut the curve in real points.*

**99. Prop. XII.** *If the diameter  $PCP'$  bisect all chords of a hyperbola parallel to the diameter  $DCD'$ , then will the diameter  $DCD'$  bisect all chords of the conic which are parallel to  $PCP'$ .*

See Art. 68.

**100. Prop. XIII.** *The lines joining any point of a hyperbola to the two extremities of any diameter are parallel to conjugate diameters.*

See Art. 69.

**101. Prop. XIV.** *If the sides of a parallelogram touch a hyperbola, its diagonals will be conjugate diameters of the hyperbola.*

See Art. 70.

**102. Measurement of lines.** In order to understand the full significance of properties of the hyperbola, regard must be paid not merely to the lengths of segments of straight lines but also to the *direction* in which these segments are measured.

The direction of any segment is shewn by the order of the letters which define its extremities. Thus  $AB$  denotes a length measured from  $A$  towards  $B$ , and  $BA$  denotes the same absolute length measured from  $B$  towards  $A$ .

When we have to consider different segments of the same straight line, all lengths measured in one direction are considered as *positive*, and consequently lengths measured in the opposite direction must be considered as *negative*. It is immaterial, and it is rarely necessary to specify, which of the two directions is taken as the positive direction.

It is at once obvious that  $AB + BA = 0$ .

The relation  $AB + BC = AC$  will also hold good for all positions of  $A$ ,  $B$  and  $C$  on a straight line.

The relation in fact only asserts that the step from  $A$  to  $B$  followed by that from  $B$  to  $C$  is equivalent to the step from  $A$  to  $C$ , and this is obviously true wherever  $B$  may be with reference to  $A$  and  $C$ .

Again, the rectangle contained by two segments of a straight line which are in the same direction is considered as *positive*, and the rectangle contained by two segments of a straight line which are in opposite directions is *negative*. Thus  $AB \cdot AC + AB \cdot CA = 0$ .

Attention to the above principles will enable us to see more clearly the correspondence between the properties of the ellipse and the hyperbola.

For example, we have proved in Art. 87 that  $tC \cdot Cn = BC^2$ ,  $tC$  and  $Cn$  being drawn in the *same* direction. Hence  $Ct \cdot Cn = -BC^2 = Cb^2$  [Art. 92]. Thus in the hyperbola, as in the ellipse,  $Ct \cdot Cn$  is equal to the square of the semi-conjugate axis.

**103.** We have already proved in Art. 24 that the ratio of the rectangles contained by the segments of any two intersecting chords of a hyperbola, which are parallel to two given straight lines respectively, is constant for all positions of the point of intersection; and, by considering lines through the centre of the hyperbola, it follows that *the ratio of the rectangles contained by the segments of any two chords of a hyperbola is equal to the ratio of the squares of the parallel semi-diameters.*

If the rectangle of the segments of one of the chords be positive [see Art. 102] and of the other be negative, one of the semi-diameters parallel to the chords will be real and the other imaginary. But if both rectangles are positive or both negative, the semi-diameters parallel to the chords must be both real or both imaginary.

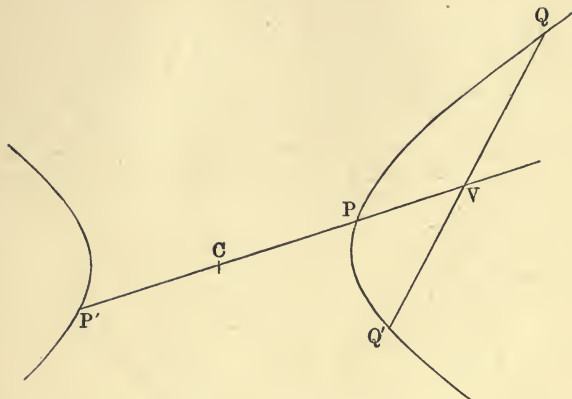
**104. Prop. XV.** *If  $QV$  be any ordinate to the diameter  $PCP'$  of a hyperbola, then will the ratio  $QV^2 : CV^2 - CP^2$  be constant.*

Let  $QV$  be produced to meet the hyperbola again in  $Q'$ ; then since  $QQ'$  is parallel to the tangent at  $P$ ,  $V$  will be the middle point of  $QQ'$ .

Now, we know [Art. 103] that the ratio of the rectangles contained by the segments of two chords of a hyperbola drawn in given directions is constant.

Thus  $VQ \cdot VQ' : VP \cdot VP'$  is constant,

i.e.  $-QV^2 : CV^2 - CP^2$  is constant.



Now, if we suppose the proposition to remain true when the chords intersect in the centre, and if  $Cd$  be the semi-diameter parallel to  $QQ'$ , which we know is of imaginary length; then we have

$$-QV^2 : CV^2 - CP^2 = Cd^2 : CP^2,$$

or 
$$QV^2 : CV^2 - CP^2 = -Cd^2 : CP^2.$$

**Conversely**, if a line  $QV$  be drawn in any fixed direction cutting the fixed line  $PP'$  in  $V$  so that the ratio  $QV^2 : VP \cdot VP'$  is constant, then the locus of the point  $Q$  will be a hyperbola or an ellipse according as the point  $V$  is without or within the terminated straight line  $PP'$ .

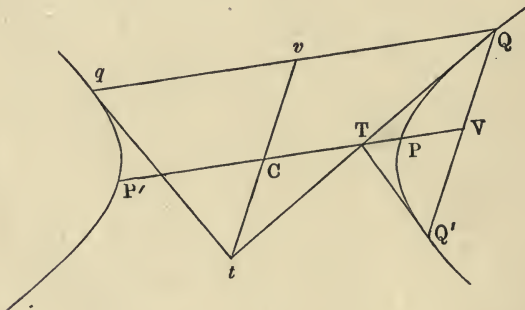
**Cor.** If on any ordinate  $QV$  to a fixed diameter  $PP'$  of a hyperbola a point  $R$  be taken such that  $VR : VQ$  is constant, the locus of  $R$  will be another hyperbola of which  $PP'$  is a diameter.

**105. Prop. XVI.** *If the tangents at the extremities of a chord  $QQ'$  of a hyperbola meet in  $T$ , and the diameter  $CT$  cut  $QQ'$  in  $V$  and the curve in  $P$ ; then will*

$$CV \cdot CT = CP^2.$$

If the diameter  $CT$  cut the curve in real points, the proof of Art. 71 is applicable without change.

If the tangents at  $Q, q$ , the extremities of a chord which cuts both branches, meet in  $t$ ,  $Ct$  will bisect  $Qq$ . And, if  $QQ'$  be parallel to  $Cv$ , the tangents at  $Q, Q'$  will meet at  $T$  on the diameter  $PCP'$  which is parallel to  $Qq$ .



Then  $vC = QV$ , and we have

$$CT : TV = Ct : QV;$$

$$\therefore CV \cdot CT : CV \cdot TV = vC \cdot Ct : QV^2.$$

Now  $CV \cdot TV = CV^2 - CV \cdot CT = CV^2 - CP^2$ , by the first case,

$$\therefore vC \cdot Ct : QV^2 = CP^2 : CV^2 - CP^2$$

$$= -Cd^2 : QV^2. \quad [\text{Prop. XV.}]$$

Hence

$$vC \cdot Ct = -Cd^2, \text{ or } Cv \cdot Ct = Cd^2.$$

**106. Prop. XVII.** *The length of a focal chord of a hyperbola varies as the square of the parallel semi-diameter.*

The proof given in Art. 79 is applicable whether the focal chord does or does not have its extremities on the same branch of the curve.

**107. Prop. XVIII.** *If a circle cut a hyperbola in four points, the line joining any two of the points and the line joining the other two points will make equal angles with the axes of the hyperbola.*

The proof given in Art. 80 is applicable without change to the hyperbola.

It should, however, be noticed that since the diameters parallel to the chords are equal they must both be real or both imaginary, from which it follows that both chords or neither will cut opposite branches of the hyperbola, and consequently that the four points of intersection of a circle and a hyperbola must all be on one branch or two on each branch of the curve.

### EXAMPLES.

1. Shew that, if a parallelogram be inscribed in a hyperbola, the diagonals of the parallelogram will be diameters of the hyperbola.

2. Shew that a circle cannot cut a hyperbola in more than four points.

3. If the chords  $PQ$ ,  $PQ'$  of a hyperbola be equally inclined to the transverse axis, the circle  $PQQ'$  will touch the hyperbola.

4. If the normal at any point  $P$  of a hyperbola cut the transverse and conjugate axes in  $G$ ,  $g$  respectively, the projections of  $PG$  and  $Pg$  on either focal distance will be respectively equal to the semi-latus-rectum, and the semi-transverse axis.

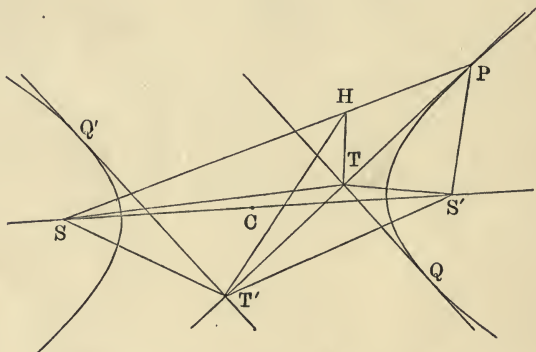
5. Shew that, if  $QQ'$  be any one of a system of parallel chords of a hyperbola, the locus of a point  $P$  on  $QQ'$  such that  $QP : PQ'$  is constant, is another hyperbola.

6. Shew that, if  $SY$ ,  $SZ$  be the perpendiculars from a focus of a hyperbola on two tangents which meet in the point  $T$ ,  $YZ$  will be perpendicular to the line joining  $T$  to the other focus of the hyperbola.

**108. Prop. XIX.** *If the tangent at any point  $P$  of a hyperbola whose foci are  $S, S'$  be cut by any pair of parallel tangents in the points  $T, T'$  and by an asymptote in the point  $L$ ; then will*

$$PT \cdot PT' = PS \cdot PS' = PL^2 = PL'^2.$$

Let  $Q, Q'$  be the points of contact of the parallel tangents. Join  $ST, S'T, ST', S'T'$ . On  $SP$  take a point  $H$ , such that  $PH = PS'$ , and join  $HT, HT'$ .



Then, since  $TT'$  bisects the angle  $HPS'$  and  $PH = PS'$ , it follows that  $TH = TS'$  and  $T'H = T'S'$ , and therefore that the triangles  $HTT', STT'$  are equal in all respects.

$$\begin{aligned} \text{Hence} \quad \angle HTT' &= \angle S'TT' \\ &= \angle STQ \end{aligned} \quad [\text{Prop. XI.}]$$

$$\therefore \angle STH = \angle T'TQ.$$

$$\text{Similarly} \quad \angle ST'H = \angle TT'Q'.$$

$$\text{But} \quad \angle T'TQ = \angle TT'Q',$$

since  $TQ$  and  $T'Q'$  are parallel ;

$$\therefore \angle STH = \angle ST'H.$$

Thus  $S, T', T, H$  lie on a circle, and therefore

$$PT \cdot PT' = PH \cdot PS = PS \cdot PS'.$$

**Cor.** Since the diagonals of a parallelogram whose sides touch a hyperbola are conjugate diameters, the proposition may be enunciated thus:

*If the tangent at  $P$  to a hyperbola be cut in  $T, T'$  by any pair of conjugate diameters ; then will*

$$PT \cdot PT' = PS \cdot PS'.$$

Since an asymptote is in the direction of two coincident conjugate diameters, it follows that *if the tangent at  $P$  cut an asymptote in  $L$  ; then  $PS \cdot PS' = PL^2$ .*

It is obvious from the above that *the portion of any tangent to a hyperbola cut off by any pair of parallel tangents, or by any pair of conjugate diameters, subtends equal angles at the foci.*

### EXAMPLES.

1. If the tangent at any point  $P$  of a hyperbola whose foci are  $S, S'$  be cut by any pair of conjugate diameters in the points  $T, T'$  respectively, then will the rectangle contained by the perpendiculars from  $T$  and  $T'$  on  $SP$  be constant.

2. If any tangent to a hyperbola be cut by the two perpendicular tangents in the points  $T, T'$ , the circle through  $T, T'$  and either focus will be equal to the auxiliary circle.

3. If any tangent to a hyperbola be cut by any pair of parallel tangents in the points  $T, T'$ , the circle through  $T, T'$  and a focus of the hyperbola will never be less than the auxiliary circle.

## PROPERTIES OF ASYMPTOTES.

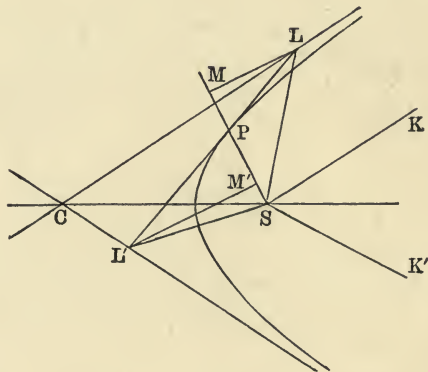
**109. Prop. XX.** *The portion of any tangent to a hyperbola intercepted between the asymptotes is bisected at its point of contact.*

Let the tangent at any point  $P$  of a hyperbola cut the asymptotes in the points  $L, L'$ .

Join  $P$  to the focus  $S$ , and draw  $SK$  parallel to the asymptote  $CL$ .

Draw from  $L$  and  $L'$  the perpendiculars  $LM, L'M'$  on  $SP$ .

Then the asymptote  $CL$  is a tangent whose point of contact is at infinity, and therefore  $LS$  makes equal angles with  $PS$  and  $SK$ . [Art. 17.]



Hence the perpendicular from  $L$  on  $PS$  is equal to the perpendicular from  $L$  on  $SK$ , and therefore equal to the perpendicular from  $S$  on  $CY$ , which is equal to  $BC$ . [Art. 94.]

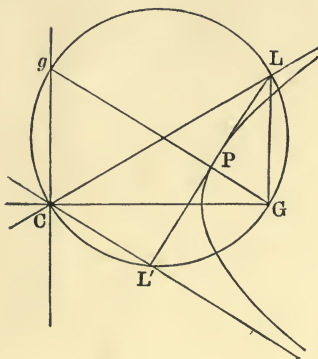
Thus  $LM = BC$ .

Similarly  $L'M' = BC$ .

Since the perpendiculars  $LM$ ,  $L'M'$ , are equal, it follows that  $LP = PL'$ .

**Ex. 1.** *The circle circumscribing the triangle formed by any tangent to a hyperbola and the two asymptotes, will pass through the points where the corresponding normal cuts the axis.*

Let any tangent cut the asymptotes in  $L$ ,  $L'$ , and let the circle  $LCL'$  cut the transverse and conjugate axes in  $G$ ,  $g$  respectively. Let  $P$  be the point of intersection of  $Gg$  and  $LL'$ .



Then  $\angle PLG + \angle PGL = \angle GCL' + \angle gCL$   
 $= \angle GCL + \angle LCg$   
 $= \text{right angle.}$

Hence  $Gg$  is perpendicular to  $LL'$ .

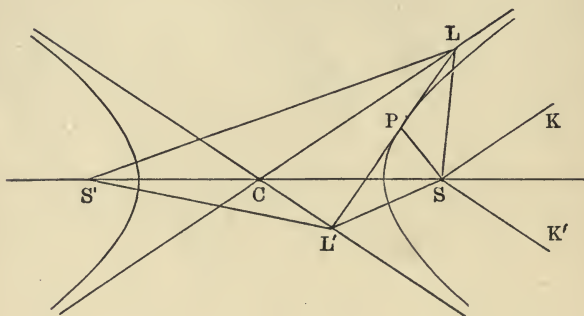
And, since the diameter  $Gg$  of the circle  $LCL'$  is perpendicular to the chord  $LL'$ , it must bisect that chord in  $P$ .

Hence  $P$  is the point of contact of the tangent  $LL'$ , and therefore  $Gg$  is the normal at  $P$ .

Ex. 2. *The portion of any tangent to a hyperbola intercepted between the asymptotes subtends supplementary angles at the foci.*

Let the tangent at  $P$  cut the asymptotes in  $L, L'$  respectively. Through  $S$  draw  $SK, SK'$  parallel to the asymptotes, and join  $SP$ .

Then  $LS$  bisects the angle  $PSK$ , and  $L'S$  bisects the angle  $PSK'$ .



$$\begin{aligned}\text{Hence} \quad 2 \angle LSL' &= \angle KSP + \angle PSK' \\ &= 2 \angle CSK = 2 \angle S'CL.\end{aligned}$$

$$\text{Thus} \quad \angle LSL' = \angle LCS'.$$

$$\text{Similarly} \quad \angle LS'L' = \angle LCS.$$

Hence the angles  $LSL', LS'L'$  are supplementary; and therefore, as  $S$  and  $S'$  are on opposite sides of  $LL'$ , **the four points  $S, S', L, L'$  are on a circle.**

Ex. 3. Shew that the triangles  $LSL'$  and  $LCS'$  are similar.

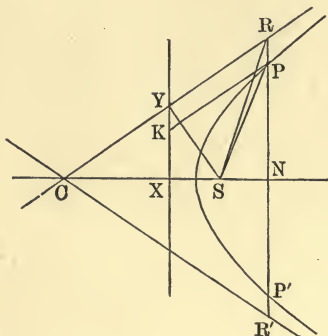
Ex. 4. Shew that the triangles  $LCS$  and  $SCL'$  are similar, and that  $LC \cdot CL' = CS^2$ .

Ex. 5. Shew that  $LS$  touches the circle  $CSL'$ , and  $L'S$  touches the circle  $CSL$ .

**110. Prop. XXI.** *If a double ordinate  $PNP'$  to the transverse axis of a hyperbola be produced to cut the asymptotes in  $R, R'$ , then will  $RP \cdot PR' = RP \cdot RP' = BC^2$ .*

Let the directrix corresponding to the focus  $S$  cut an asymptote  $CR$  in  $Y$ ; then we know that  $SY$  is perpendicular to the asymptote.

Draw  $PK$  parallel to the asymptote  $CYR$  to meet the directrix in  $K$ ; then we know that  $SP = PK$ , and therefore  $SP = RY$ .



Since  $RR'$  and  $PP'$  are both bisected in  $N$ , it is obvious that  $RP = P'R'$  and  $RP' = PR$ .

$$\begin{aligned}
 \text{Hence } RP \cdot PR' &= RP \cdot RP' = RN^2 - PN^2 \\
 &= SR^2 - SP^2 \\
 &= SR^2 - RY^2 \\
 &= SY^2 = BC^2.
 \end{aligned}$$

**Cor.** Since the rectangle  $RP \cdot PR'$  is invariable, and  $PR'$  increases without limit as  $CN$  becomes greater and greater, it follows that  $RP$  diminishes without limit as  $CN$  is increased. Thus the asymptote approaches indefinitely near to the curve but never crosses it.

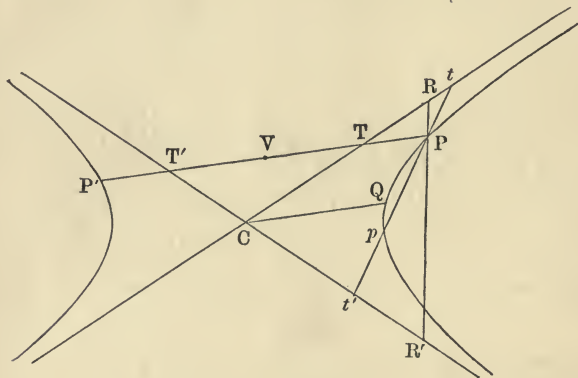
**111. Prop. XXII.** *If a straight line cut a hyperbola in the points  $P, P'$  and the asymptotes in the points  $T, T'$ ; then will the rectangle  $PT.PT'$  be equal to the square of the parallel semi-diameter of the hyperbola, and  $PT$  will be equal to  $T'P'$ .*

Through  $P$  draw a straight line perpendicular to the transverse axis and cutting the asymptotes in  $R, R'$  respectively.

Then we know [Prop. XXI.] that the rectangle  $PR.PR'$  is constant and equal to the square of the parallel semi-diameter.

But if the chord  $PTT'P'$  be drawn in any given direction, each of the triangles  $RPT, R'PT'$  is of invariable form; and therefore  $PR:PT$  is constant, and  $PR':PT'$  is constant for all positions of  $P$ .

Hence the rectangles  $PT.PT'$  and  $PR.PR'$  are in a constant ratio, and we know that the rectangle  $PR.PR'$  is constant.



The rectangle  $PT.PT'$  is therefore constant for all positions of  $P$  provided  $PTT'$  is drawn in a given direction.

If the chord be drawn through the point  $Q$  on the hyperbola, such that the diameter  $QC$  is parallel to  $PTT'$ , it follows that

$$PT \cdot PT' = QC^2.$$

If the chord  $PTT'$  cut different branches of the hyperbola, the parallel diameter will cut the curve in real points.

If, however, a line be drawn through  $P$  which cuts the curve in the two points  $P, p$  on the same branch of the curve and which cuts the asymptotes in  $t, t'$  respectively, the parallel diameter will cut the curve in *imaginary* points, but the rectangle  $Pt \cdot Pt'$  will still be equal to the square of the parallel semi-diameter, the rectangle  $Pt \cdot Pt'$  and the square of the parallel semi-diameter being both *negative* in this case.

Again, if the straight line  $PTT'$  cut the hyperbola again in  $P'$ , and  $V$  be the middle point of  $TT'$ ; then

$$PT \cdot PT' = P'T \cdot P'T';$$

$$\therefore PV^2 - TV^2 = P'V^2 - T'V^2.$$

Hence, as  $TV = VT'$ ,  $PV = VP'$ . Thus, *the middle point of  $TT'$  is also the middle point of  $PP'$* ; and therefore  $PT = T'P'$ .

As a particular case, when the chord is such that the points  $P, P'$  coincide, the middle point of  $TT'$  will coincide with either  $P$  or  $P'$ .

This gives another proof of Prop. XX.

**Cor.** If the tangent at  $P$  cut the asymptotes in  $L, L'$  and  $Cd$  be the parallel semi-diameter,

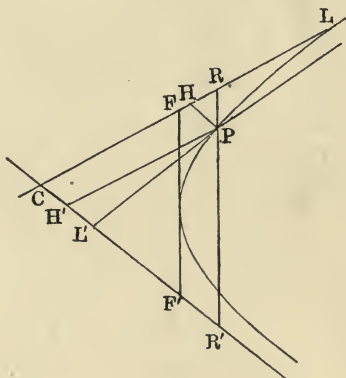
$$Cd^2 = PL \cdot PL' = -PL^2,$$

since

$$LP = PL'.$$

**112. Prop. XXIII.** *The triangle formed by the asymptotes and any tangent to a hyperbola is of constant area.*

Let the tangent at  $P$  cut the asymptotes in  $L, L'$ . Draw  $PH, PH'$  parallel to  $CL', CL$  respectively, and meeting  $CL, CL'$  in  $H, H'$  respectively. Draw also through  $P$  a line perpendicular to the transverse axis cutting  $CL, CL'$  in  $R, R'$  respectively.



Since each of the triangles  $HPR, H'PR'$  is of invariable form it follows that  $PH:PR$  and  $PH':PR'$  are both constant, and therefore that  $PH.PH':RP.PR'$  is constant.

But we know that  $RP.PR' = BC^2$ . [Prop. XXI.]

Hence  $PH.PH'$  is constant.

But, since  $LL'$  is bisected in  $P$  and  $PH', PH$  are parallel to  $CL, CL'$  respectively,  $CL' = 2HP$  and  $CL = 2H'P$ .

Hence  $CL.CL' = 4PH.PH' = \text{constant}$ ,  
and therefore the triangle  $LCL'$  is of constant area.

Now, if the tangent at a vertex cut the asymptotes in  $F, F'$  respectively, we know that

$$CF = CF' = CS.$$

Hence  $CL \cdot CL' = CF \cdot CF' = CS^2.$

Also  $4PH \cdot PH' = CL \cdot CL' = CS^2.$

**Otherwise thus:—**

Since  $C$  is the middle point of  $SS'$ ,

$$\begin{aligned} 2\Delta LCL' &= \Delta LS'L' - \Delta LL'S \\ &= 2\Delta LPS' - 2\Delta LPS \\ &= S'P \cdot BC - SP \cdot BC \quad [\text{Art. 109}] \\ &= 2AC \cdot BC. \end{aligned}$$

**Or thus:—**

The points  $L, L', S, S'$  are cyclic. [Art. 109, Ex. 2.]

Let then the circle  $LSL'S'$  cut the asymptote  $CL$  again in the point  $l$ .

Then, since  $C$  is the middle point of  $SS'$ , and  $CL', Cl$  make equal angles with  $SS'$  and are on the same side of it, it follows that  $Cl = CL'.$

Hence  $CL \cdot CL' = CL \cdot Cl = SC \cdot CS'.$

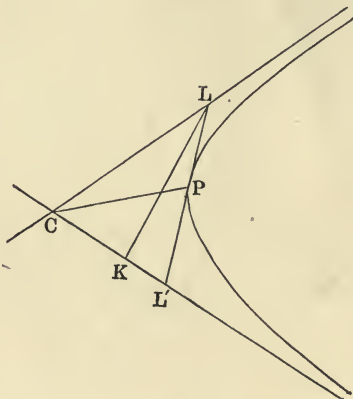
### EXAMPLES.

1. Construct a hyperbola when the asymptotes and one point on the curve are given.
2. Construct the hyperbola of which one asymptote and three points are given.
3. Construct a hyperbola having given one asymptote, two points on the curve and the tangent at one of these points.
4. Shew that, if a line move in such a manner that the triangle formed by it and two fixed lines is of constant area, the line will always touch a fixed hyperbola. Prove also that the locus of a point which divides in a given ratio the part of the moving line intercepted between the fixed lines, is also a hyperbola of which the fixed lines are asymptotes.
5. Shew that, if two tangents to a hyperbola cut the asymptotes in  $L, L'$  and  $M, M'$  respectively; then will  $LM'$  and  $L'M$  be parallel to the chord of contact of the tangents and equidistant from it.
6. Construct a hyperbola having given one asymptote, two tangents and the point of contact of one of them.

**113. Prop. XXIV.** *The sum of the squares of conjugate diameters of a hyperbola is constant.*

Let the tangent at any point  $P$  of a hyperbola cut the asymptotes in  $L, L'$  respectively.

Then we know that  $P$  is the middle point of  $LL'$ .



$$\text{Hence} \quad CL^2 + CL'^2 = 2CP^2 + 2PL^2.$$

Also, if  $LK$  be the perpendicular from  $L$  on  $CL'$ ,

$$CL^2 + CL'^2 - 2CL' \cdot CK = LL'^2 = 4PL^2.$$

$$\text{Hence} \quad CL' \cdot CK = CP^2 - PL^2.$$

But we know that  $CL \cdot CL'$  is constant, and the triangle  $LCK$  being of invariable form,  $CK$  will vary as  $CL$ , so that the rectangle  $CL' \cdot CK$  is constant. Thus  $CP^2 - PL^2$  is constant. But [Prop. XXII., Cor.]

$$PL^2 = -Cd^2,$$

where  $Cd$  is the semi-diameter parallel to  $LPL'$ . Hence  $CP^2 + Cd^2$  is constant.

**114.** When a pair of conjugate diameters of a hyperbola are given, the axes, foci, &c. can be determined.

For let  $PCP'$  be the given diameter which cuts the curve in real points, and let  $KCK'$  be the direction of the conjugate diameter. Then the diameter  $KCK'$  will cut the curve in imaginary points  $d, d'$  such that  $-Cd^2$  is equal to a given square.

Then the line through  $P$  parallel to  $KCK'$  will be the tangent at  $P$ ; and, if the points  $L, L'$  be taken on this tangent equidistant from  $P$  and such that

$$LP^2 = PL^2 = -Cd^2,$$

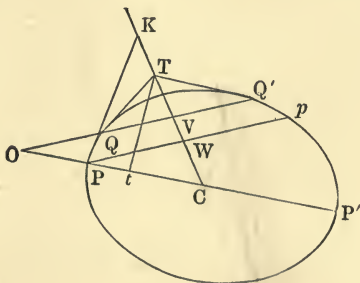
then  $L, L'$  will be on the asymptotes. Thus the asymptotes  $CL, CL'$  are found, and the axes are the bisectors of the angle  $LCL'$ , the transverse axis being along the bisector which is in the same compartment as the point  $P$ .

Now take points  $K, K'$  on the asymptotes such that

$$CK^2 = CK'^2 = CL \cdot CL';$$

then  $KK'$  is the tangent at a vertex of the hyperbola, and  $CK = CS$ . Thus the circle centre  $C$  and radius  $CK$  will cut the transverse axis in the foci.

**115. Prop. XXV.** *If any line be drawn through a fixed point to cut a conic, the tangents at the points of intersection will meet on a fixed straight line.*



Let  $QQ'$  be any chord through the fixed point  $O$ , and let the tangents at  $Q, Q'$  meet in  $T$ ; then  $TC$  will bisect  $QQ'$  in  $V$ . Draw  $Tt$  parallel to the diameter conjugate to  $CO$  and cutting  $CO$  in  $t$ .

First, if  $CO$  cut the conic in real points  $P, P'$ . Let  $Pp$  be drawn parallel to  $QQ'$ , and let the tangents at  $P, p$  meet at  $K$ ; then  $K$  will be on the diameter  $CT$ , and  $CV \cdot CT = CK \cdot CW$ , where  $W$  is the point of intersection of  $Pp$  and  $CT$ .

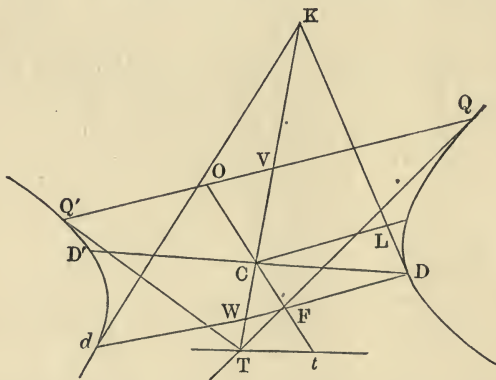
Then, since  $Tt$  is parallel to  $PK$ , we have

$$\begin{aligned} Ct : Cp &= CT : CK \\ &= CW : CV, \text{ since } CV \cdot CT = CK \cdot CW \\ &= CP : CO, \text{ since } PW \text{ and } OV \text{ are parallel.} \end{aligned}$$

Hence  $CO \cdot Ct = CP^2$ , and therefore  $t$  is a fixed point.

Next, if  $CO$  do not cut the curve in real points, the diameter conjugate to  $CO$  will cut the curve in real points,  $D, D'$  suppose.

Let  $Dd$  be a chord parallel to  $QQ'$  cutting  $TC$  in  $W$  and  $OCt$  in  $F$ ; then the tangents at  $D, d$  will meet in a point  $K$  on  $TC$  such that  $WC \cdot CK = TC \cdot CV$ . [Art. 105.]



Let the tangent at  $D$  cut the diameter parallel to  $QQ'$  in  $L$ .

Then, since  $CL$ ,  $CK$  are conjugate diameters, the rectangle  $DL \cdot DK$  is constant for all directions of the chord  $QQ'$ . [Art. 108.]

$$\begin{aligned}\text{Then } FC : CO &= WC : CV \\ &= CT : KC, \text{ since } CT \cdot CV = CW \cdot CK, \\ &= Ct : KD,\end{aligned}$$

since the triangles  $TCt$  and  $CKD$  have their sides parallel and are therefore similar.

Hence  $CO \cdot Ct = FC \cdot KD = -DL \cdot DK = \text{constant}$ .

Hence  $t$  is a fixed point, and therefore  $T$  lies on a fixed straight line.

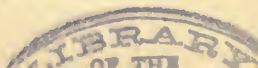
**Def.** The straight line which is the locus of the point of intersection of the tangents at the extremities of any chord of a conic drawn through a fixed point, is called the **polar** of the point, and the point is called the **pole** of the line, with respect to the conic.

The converse theorem can be easily deduced, namely, that *if any point be taken on a given straight line and tangents be drawn from it to a conic, the line joining the points of contact of the tangents will pass through a fixed point.*

If the point  $O$  be without the conic, two lines can be drawn through  $O$  each of which will cut the conic in coincident points, namely, the two tangents from  $O$  to the conic; and when the points  $Q$ ,  $Q'$  coincide the point of intersection of the tangents at  $Q$ ,  $Q'$  will coincide with them. Thus *when a point  $O$  is without a conic, the tangents at the extremities of any chord through  $O$  will meet on the line through the points of contact of the tangents from  $O$  to the conic.*

**Cor. I.** *The polar of a fixed point with respect to a conic will or will not cut the conic in real points according as the fixed point is without or within the conic.*

**Cor. II.** *If the polar of a point  $A$  with respect to a conic pass through the point  $B$ , then will the polar of  $B$  pass through  $A$ .*

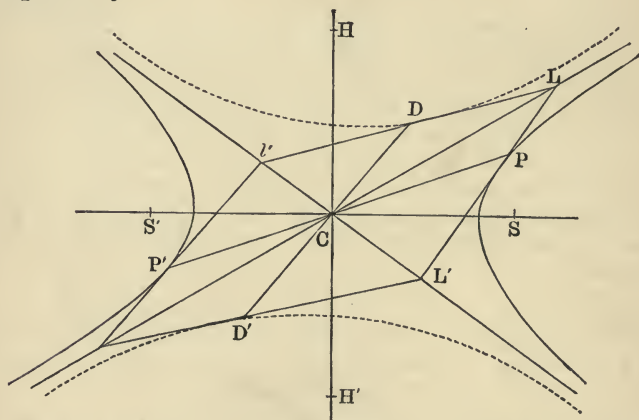


## THE CONJUGATE HYPERBOLA.

**116. Def.** Two hyperbolas which have the same asymptotes and whose foci are at the same distance from their common centre are called **conjugate hyperbolas**.

Since the axes of a hyperbola bisect the angles between the asymptotes it follows that the axes of two conjugate hyperbolas coincide, the transverse axis of one curve must however lie along the conjugate axis of the other.

Let  $P$  be any point on a hyperbola whose foci are  $S, S'$ , and let the tangent at  $P$  cut the asymptotes in  $L, L'$  respectively.



From  $L$  draw the tangent  $LDl'$  touching the conjugate hyperbola in  $D$ , and cutting the asymptote  $CL'$  in  $l'$ .

Then we know that  $D$  will be the middle point of  $Ll'$  and that  $CL \cdot Cl' = CH^2$ , where  $H$  is a focus of the conjugate hyperbola.

$$\text{Hence } LC \cdot Cl' = CH^2 = CS^2 = LC \cdot CL';$$

$$\therefore L'C = Cl'.$$

But  $LD = Dl'$  and  $LP = Pl'$ ; therefore  $CD$  is parallel to  $LPL'$  and  $CP$  is parallel to  $LDl'$ .

Thus conjugate hyperbolas have coincident conjugate diameters, and the diameter which cuts one curve in real points lies along the direction of the diameter which cuts the other curve in imaginary points.

Since  $PCDL$  is a parallelogram,  $PD$ , and similarly  $P'D$ , is parallel to one asymptote and is bisected by the other.

The area of the parallelogram formed by the tangents at the extremities of the diameter  $PCP'$  of one hyperbola and the tangents at the extremities of the conjugate diameter of the conjugate hyperbola is clearly four times the area of the triangle  $LCL'$ ; and is therefore constant.

Since  $CD = PL$ , it follows from Art. 113 that  $CP^2 - CD^2$  is constant.

Thus the difference of the squares of any diameter of a hyperbola and the conjugate diameter of the conjugate hyperbola is constant.

#### THE RECTANGULAR HYPERBOLA.

**117.** If a directrix cut the asymptotes  $CY$ ,  $CY'$  of a rectangular hyperbola in the points  $Y$ ,  $Y'$  respectively; then, if  $S$  be the corresponding focus we know that  $SYC$  and  $SY'C$  are right angles [Art. 94], whence it follows that the figure  $SYCY'$  is a square and  $CS^2 = 2CY^2 = 2CA^2$ .

Hence the eccentricity of a rectangular hyperbola is  $\sqrt{2}$ .

**118. Prop. XXVI.** Conjugate diameters of a rectangular hyperbola make equal angles with an asymptote.

Let the tangent at any point  $P$  cut the asymptotes in the points  $L$ ,  $L'$ .

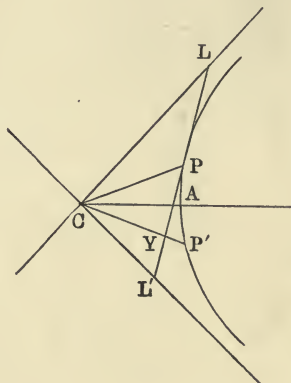
Since the angle  $LCL'$  is a right angle, and  $P$  is the middle point of  $LL'$  [Art. 109], it follows that  $P$  is the centre of the circle  $LCL'$  and that the angles  $PCL$  and  $PLC$  are equal.

Thus, if  $Cd$  be the diameter conjugate to  $CP$ , and therefore parallel to  $LPL'$ ,  $CP$  and  $Cd$  will make equal angles with either asymptote.

**Cor.** The angles between any two diameters, or any two chords, of a rectangular hyperbola are equal or supplementary to the angles between their conjugate diameters.

**119. Prop. XXVII.** *The sum of the squares of two conjugate diameters, or of two perpendicular diameters, of a rectangular hyperbola, is zero.*

Let the tangent at any point  $P$  cut the asymptotes in  $L, L'$ .



Then we know that the square of  $Cd$ , the semi-diameter conjugate to  $CP$ , is equal to  $-PL^2$ . [Prop. XXII.]

But  $LCL'$  is a right angle and  $P$  is the middle point of  $LL'$ ; hence  $PL = CP$ .

$$\text{Therefore } CP^2 + Cd^2 = CP^2 - PL^2 = 0.$$

Next, let  $CP'$  be the semi-diameter such that the angles  $P'CA$  and  $ACP$  are equal; and let  $CP'$  cut  $LPL'$  in  $Y$ .

$$\begin{aligned} \text{Then } \angle YCP + \angle YPC &= 2\angle ACP + 2\angle PCL \\ &= 2\angle ACL = \text{a right angle.} \end{aligned}$$

Hence  $CP'$  is perpendicular to  $LPL'$ .

Since  $CP$  and  $CP'$  are equally inclined to the transverse axis,  $CP = CP'$ .

$$\text{Hence } CP'^2 + Cd^2 = CP^2 + Cd^2 = 0.$$

Since there are only two pairs of equal diameters of any conic, which we know are equally inclined to an axis, it follows that, *if the sum of the squares of two diameters of a rectangular hyperbola be zero, the diameters must be conjugate or perpendicular.*

**Conversely.** *If a conic be such that (1) the sum of the squares of a pair of conjugate diameters is zero, or (2) if the sum of the squares of two perpendicular diameters is zero; then in either case the conic must be a rectangular hyperbola.*

The conic must be a *hyperbola*, since the length of one diameter is real and of the other is imaginary.

Let  $P$  be any extremity of the real diameter, and let the tangent at  $P$  cut the asymptotes in  $L, L'$ .

Then in the first case, since

$$CP^2 = -Cd^2 = PL^2, \quad CP = LP = PL',$$

whence  $LCL'$  must be a right angle.

To prove the second case, if  $CP$  be the real diameter, and if  $KPK'$  be drawn perpendicular to  $CP$  so as to cut the asymptotes of the hyperbola in  $K, K'$  respectively,

Then we know that  $PK.PK'$  is equal to  $Cd^2$ , the square of the semi-diameter parallel to  $KPK'$ . Thus  $KP.PK' = -Cd^2 = CP^2$ , by supposition.

And, since  $CP$  is perpendicular to  $KPK'$  and

$$KP.PK' = CP^2,$$

it follows that  $KCK'$  is a right angle.

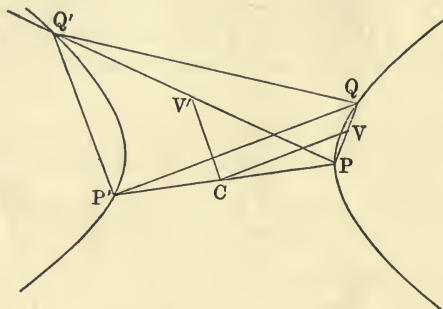
Ex. 1. Construct a rectangular hyperbola having given the centre and any two points on the curve.

Ex. 2. Construct a rectangular hyperbola having given one asymptote and any two points on the curve.

Ex. 3. Construct a rectangular hyperbola having given two points on the curve and the tangents at those points.

**120. Prop. XXVIII.** *Any chord of a rectangular hyperbola subtends equal or supplementary angles at the extremities of any diameter.*

Let  $Q, Q'$  be the extremities of any chord, and  $PCP'$  be any diameter of a rectangular hyperbola.



Join  $QP, Q'P, QP', Q'P'$ . Bisect  $PQ$  in  $V$  and  $PQ'$  in  $V'$ , and join  $CV, CV'$ .

Then the angle  $QPQ'$  between the chords  $PQ$  and  $PQ'$  is equal, or supplementary, to the angle  $VCV'$  between the conjugate diameters  $CV$  and  $CV'$ .

But, since  $PQ$  is bisected in  $V$  and  $PP'$  in  $C$ ,  $CV$  must be parallel to  $P'Q$ . Similarly  $CV'$  must be parallel to  $P'Q'$ . Hence  $\angle QP'Q' = \angle VCV'$ .

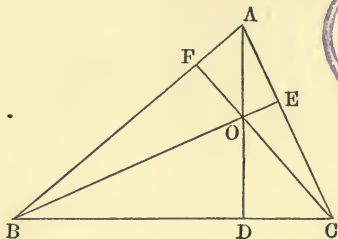
Therefore the angles  $QP'Q'$  and  $QPQ'$  are either equal or supplementary.

**Cor.** *The locus of a point  $C$  which moves so that the difference of the angles  $CBA$  and  $CAB$  is constant,  $A$  and  $B$  being fixed points, is a rectangular hyperbola of which  $AB$  is a diameter.*

For, if the rectangular hyperbola which has  $AB$  as a diameter and passes through any one position of the moving point be constructed, it is easy to shew that every other position of the moving point is on this hyperbola.

**121. Prop. XXIX.** *If a rectangular hyperbola pass through the angular points of a triangle it will also pass through the orthocentre; and conversely every conic which passes through the angular points and the orthocentre of a triangle must be a rectangular hyperbola.*

Let  $AD$ ,  $BE$ ,  $CF$  be the perpendiculars of the triangle  $ABC$ , and let  $O$  be the orthocentre.



Then it is easily seen that the rectangles  $BD \cdot DC$  and  $DO \cdot DA$  are equal.

Let a rectangular hyperbola through  $A$ ,  $B$ ,  $C$  be cut again by  $DA$  in the point  $P$ . Then, since the sum of the squares of perpendicular diameters is zero, we must have

$$DP \cdot DA = -DB \cdot DC = BD \cdot DC.$$

Thus  $DO \cdot DA = DP \cdot DA$ ,

so that  $P$  must coincide with  $O$ .

Conversely, if a conic pass through  $A$ ,  $B$ ,  $C$  and  $O$ , it must be a rectangular hyperbola.

For since  $DO \cdot DA = -DB \cdot DC$ ,

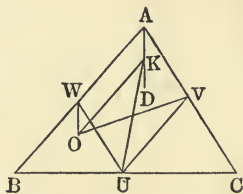
it follows that the sum of the squares of a pair of perpendicular diameters is zero, which can only be the case when the conic is a rectangular hyperbola. [Prop. XXVII.]

**Cor.** *All conics through the points of intersection of two rectangular hyperbolas are rectangular hyperbolas.*

**122. Prop. XXX.** *The locus of the centres of rectangular hyperbolas which circumscribe a triangle is the nine point circle of the triangle.*

Let  $A, B, C$  be the angular points of the triangle, and let  $D$  be its orthocentre. Then we know that every rectangular hyperbola through  $A, B$  and  $C$  will also pass through  $D$ .

Let  $U, V, W, K$  be the middle points of  $BC, CA, AB$  and  $AD$  respectively.



Then, since the angle between any two chords is equal or supplementary to the angle between their conjugate diameters, if  $O$  be the centre of one of the rectangular hyperbolas, the angle  $VOW$  is equal or supplementary to the angle  $CAB$  which is equal to  $VUW$ .

Hence  $O$  is either on the circle  $VUW$ , which is the nine point circle of  $ABC$ , or on the circle  $VAW$ .

Similarly  $O$  must be on the circle  $KUW$ , which is the nine point circle, or on the circle  $KAW$ .

The centre  $O$  must therefore be on the nine point circle of the triangle  $ABC$ ; for it cannot be at the point  $A$ .

**Cor.** The angle  $CAB$  is equal or supplementary to the angle  $VOW$  between the conjugate diameters according as  $A$  and the centre of the curve are on opposite sides or on the same side of  $VW$ .

**Ex. 1.** Find the centre of the rectangular hyperbola which passes through four given points.

**Ex. 2.** Shew that the nine point circles of the four triangles determined by taking three out of four given points meet in a point.

## LIMITING FORMS OF CONICS.

**123.** We have hitherto considered that the focus of a conic was at a finite distance from the directrix: this may not, however, be the case.

When the focus is *on* the directrix, and the eccentricity is greater than unity, it is easy to see that the conic is two straight lines through the focus, and that these straight lines will become more and more nearly coincident as the eccentricity becomes more and more nearly equal to unity.

*Thus two intersecting straight lines may be considered as a hyperbola whose foci and centre are at the intersection of the lines, and whose directrix is either of the bisectors of the angles between them; also two coincident straight lines may be considered as a parabola.*

It should be noticed that a circle is a conic whose directrix is at infinity, whose foci coincide with the centre of the circle, and whose eccentricity is zero; also that two parallel straight lines may be considered as a parabola whose focus and directrix are both at infinity.

## CONFOCAL CONICS.

**124.** Conics whose foci coincide are called **confocal conics**.

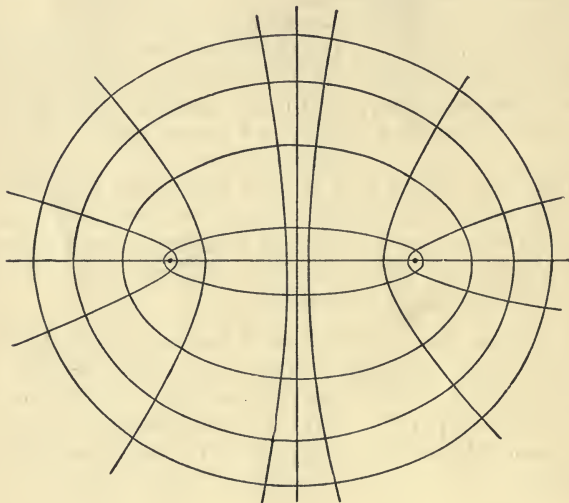
It is obvious that confocal conics have the same centre and axes. We proceed to consider other important properties of confocal conics.

When the transverse axis of an ellipse whose foci are given is very great the curve approximates to a circle of infinite radius.

As the transverse axis becomes more and more nearly equal to the distance between the given foci, the curve becomes flatter and flatter, and the line-ellipse joining the foci is a limiting form of one of the confocals.

When the transverse axis is less than the distance between the foci the conic will be a hyperbola.

As the transverse axis diminishes indefinitely and ultimately vanishes, the two branches of the hyperbola become more and more nearly coincident with the conjugate axis, and thus *the double line coincident with the conjugate axis is a limiting form of one of the confocals.*



Again, as the transverse axis increases and ultimately becomes equal to the distance between the foci, the hyperbola becomes flatter and flatter, and thus *a double line coincident with the complement of the line joining the foci is a limiting form of one of the confocals.*

**I.** *Two conics of a confocal system pass through a given point, of which one is an ellipse and the other a hyperbola.*

For having given the foci  $S, S'$  and one point  $P$  of an ellipse, the centre  $C$  is the middle point of  $SS'$ , and the major axis is equal to  $SP + S'P$ . Hence the vertices  $A, A'$  are found; and  $X, X'$ , the feet of the directrices, are given by the relation  $CS \cdot CX = CA^2$ ; also the eccentricity is equal to the ratio  $CS : CA$ . Thus the ellipse is completely determined, and only *one* ellipse can be drawn.

Similarly one and only one hyperbola with  $S, S'$  as foci will pass through the point  $P$ .

**II.** *One conic of a confocal system and only one will touch a given straight line.*

Let  $S, S'$  be the foci and  $C$  the middle point of  $SS'$ . Draw  $SY$ , the perpendicular on the given straight line. Then  $CY$  is equal to the semi-transverse axis of the conic, which is therefore completely determined.

**III.** *Two confocal conics cut one another at right angles at all their common points.*

For, if two confocal conics intersect at a point  $P$ , one must be an ellipse and the other a hyperbola. The tangent to the hyperbola bisects the angle between  $PS$  and  $PS'$ , and the tangent to the ellipse bisects the angle between  $SP$  and  $S'P$  produced: the tangents to the two confocals are therefore at right angles.

**IV.** *If a tangent to one of two given confocal conics be perpendicular to a tangent to the other, their point of intersection will lie on a fixed circle.*

Let  $T$  be the point of intersection of the two tangents  $TP, TQ$ ; and let  $CA, CA'$  be the semi-transverse axes of the conics on which  $P, Q$  respectively lie.

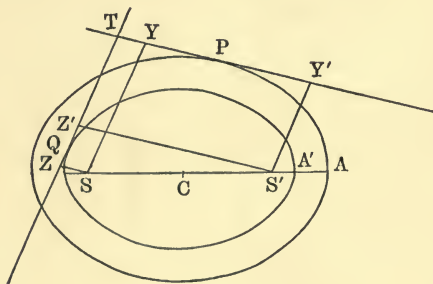
Draw  $SY, S'Y'$  perpendicular to  $TP$ ; and  $SZ, S'Z'$  perpendicular to  $TQ$ . Then, the points  $Z, Z'$  are on the auxiliary circle of the conic on which  $Q$  lies.

Hence

$$TZ \cdot TZ' = CT^2 - CA'^2.$$

But

$$SY = ZT \text{ and } S'Y' = Z'T;$$



$$\therefore TZ \cdot TZ' = SY \cdot S'Y' = CA^2 - CS^2.$$

Hence

$$CT^2 = CA^2 + CA'^2 - CS^2.$$

Thus  $T$  is on the circumference of a circle concentric with the conics.



Let  $QQ'$  be any chord of a conic and let the tangents at  $Q, Q'$  intersect in  $T$ ; and let  $CT$  cut  $QQ'$  in  $V$  and the curve in  $P$ .

Draw  $CK$  perpendicular to the tangent at  $P$  and let it cut  $QQ'$  in  $L$ .

Then, if  $QQ'$  touches a fixed confocal conic, we know that  $CK^2 - CL^2$  is constant.

But, if  $CD$  be the semi-diameter parallel to  $QQ'$ ,

$$\begin{aligned} QV^2 : CD^2 &= CP^2 - CV^2 : CP^2 \\ &= CK^2 - CL^2 : CK^2; \\ \therefore QV^2 : CK^2 - CL^2 &= CD^2 : CK^2 \\ &= CD^4 : CD^2 \cdot CK^2. \end{aligned}$$

But  $CD \cdot CK = AC \cdot BC,$

and  $CK^2 - CL^2$  is constant if  $QQ'$  touches a fixed confocal.

Hence  $QV$  varies as  $CD^2$ .

Conversely, if  $QV$  varies as  $CD^2$ , it follows from the above that  $CK^2 - CL^2$  is constant, and therefore that the chord  $QQ'$  touches a fixed confocal.

**Cor.** If  $QQ'$ , a chord of a conic, touches a fixed confocal conic, and  $Ct$  be drawn parallel to the chord  $QQ'$  to meet the tangent at  $Q$  in  $t$ ; then will  $Ct$  be of constant length.

For  $Ct \cdot QV = CD^2$ .

**VII.** The locus of the poles of a given straight line with respect to one of a system of confocal conics is a straight line perpendicular to the given straight line.

If the given straight line cut the transverse axis in  $T$ , and if a perpendicular line through its pole with respect to any conic whose foci are  $S, S'$  and centre  $C$  cut the axis in  $G$ ; then we know [Art. 82] that  $CG \cdot CT = CS^2$ .

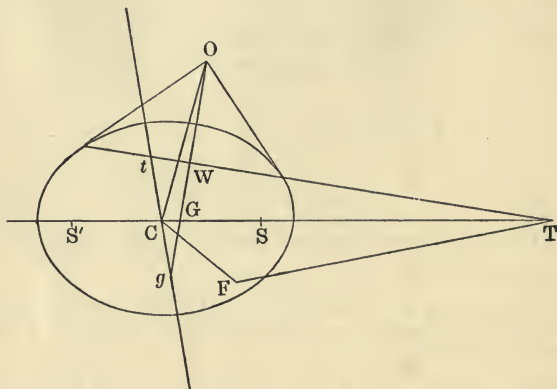
Hence  $G$  is a fixed point, and therefore the pole of the given line must lie on a perpendicular line through the fixed point  $G$ .

The given straight line will touch one of the conics of the system, and if  $P$  be the point of contact, the pole of the line with respect to the conic it touches will be the point  $P$ .

Thus the locus of the poles of a given straight line with respect to the conics of a confocal system is a perpendicular straight line which is a normal to the confocal which touches the given straight line.

**VIII.** The envelope of the polar of a given point with respect to a system of confocals is a parabola which touches the axes of the confocals.

Let  $O$  be the fixed point, and let its polar with respect to any conic of the system cut the axes in  $T, t$  respectively. Also let the line through  $O$  perpendicular to its polar cut the axes in  $G, g$  respectively and the polar in  $W$ .



Take a line  $CF$  such that  $CT$  bisects the angle  $OCF$ , and take the point  $F$  on this line such that  $CF \cdot CO = CS^2$ ; then  $F$  is a fixed point for all the conics of the system.

Since  $CF \cdot CO = CS^2$ , and  $CG \cdot CT = CS^2$  [Art. 82],

$$CG : CO = CF : CT;$$

also the angles  $GCO$  and  $FCT$  are equal.

Hence the triangles  $GCO$  and  $FCT$  are similar, and

$$\angle CFT = \angle CGO$$

$$= \text{supplement of } CtT,$$

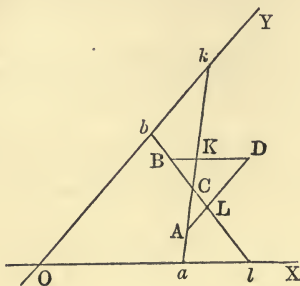
since the angles  $tCG$  and  $tWG$  are right angles.

Hence the points  $t, C, T, F$  are cyclic.

And, since the circle  $tCT$  goes through the *fixed* point  $F$ , it follows that the line  $tT$  always touches the parabola whose focus is  $F$  and of which  $CT, Ct$  are tangents.

**125.** We know that the locus of the centres of conics which touch two given straight lines at given points is a straight line. The following is an extension of this theorem:—

*The locus of the centres of conics with respect to which the poles of two given straight lines are given points, is a straight line through the intersection of the given straight lines.*



Let  $OX$ ,  $OY$  be the given straight lines, and  $A$ ,  $B$  the given points. Let  $C$  be the centre of one of the conics so that  $A$  is the pole of  $OX$ , and  $B$  of  $OY$  with respect to the conic centre  $C$ .

Let  $CA$  cut  $OX$  in  $a$ , and  $CB$  cut  $OY$  in  $b$ .

Since  $A$  is the pole of  $OX$ , and  $B$  is the pole of  $OY$ ,  $O$  will be the pole of  $AB$ .

Hence  $CO$  and  $AB$  are parallel to conjugate diameters of the conic.

Let  $AD$ ,  $BD$  be drawn parallel to  $OY$ ,  $OX$  respectively. Also let  $CA$ ,  $CB$  cut  $OY$ ,  $OX$  in  $k$ ,  $l$  respectively.

Then, since  $BD$  is parallel to  $OX$ , the pole of  $BD$  is on  $CA$ , and it is also on the polar of  $B$ ; hence  $k$  is the pole of  $BD$ .

Similarly  $l$  is the pole of  $AD$ .

Hence, if  $Ck$ ,  $Cl$  cut  $BD$ ,  $AD$  respectively in  $K$ ,  $L$ , we have

$$CK \cdot Ck = CA \cdot Ca \text{ and } CL \cdot Cl = CB \cdot Cb.$$

$$\text{Therefore} \quad Ck : CA = Ca : CK$$

$$= Cl : CB,$$

since  $BK$  and  $al$  are parallel.

Hence  $kl$  is parallel to  $AB$ .

But  $k$  is the pole of  $BD$  and  $l$  is the pole of  $AD$ , and therefore  $D$  is the pole of  $kl$ .

Hence  $CD$  and  $kl$  are parallel to conjugate diameters, and therefore  $CD$  and  $AB$  are parallel to conjugate diameters.

But we have also proved that  $CO$  and  $AB$  are parallel to conjugate diameters.

The lines  $CO$  and  $CD$  therefore coincide, so that  $C$  must lie on the fixed line  $OD$ .

## EXAMPLES.

1. From a fixed point  $S$  a line  $SP$  is drawn to meet a fixed circle in  $P$ , and  $PQ$  is drawn making an angle  $SPQ$  of given magnitude: shew that  $PQ$  envelopes a conic one focus of which is at  $S$ , and find the position of the other focus.

2. From any point  $P$  on a hyperbola a line is drawn parallel to one asymptote to meet the second asymptote in  $M$ , and from a point  $Q$  a line is drawn parallel to the second asymptote to meet the first in  $N$ . Shew that  $MN$  is parallel to  $PQ$ .

3. The tangent at the point  $P$  of a hyperbola meets an asymptote in  $T$ ,  $TQR$  is drawn parallel to the other asymptote meeting the curve in  $Q$  and the parallel through  $P$  to the first asymptote in  $R$ . Shew that  $TR$  is bisected in  $Q$ .

4. A tangent to a hyperbola at any point  $P$  meets an asymptote in  $T$ . A line  $RPR'$  is drawn parallel to this asymptote and meets a directrix in  $R'$  and  $ST$  in  $R$ , where  $S$  is the focus corresponding to the directrix. Shew that  $P$  is the middle point of  $RR'$ .

5. A conic of given transverse axis has one focus at the focus of a given parabola and touches the parabola; shew that the conic also touches another parabola coaxial and confocal with the given parabola.

6. Conics are drawn with a common focus and with transverse axes equal to a given straight line. Shew that the conics all touch two fixed conics.

7. Find the centre and axes of a rectangular hyperbola having given a focus, one asymptote and another tangent.

8. Describe the hyperbolas which have a given focus, pass through a given point and have their asymptotes parallel to two given straight lines.

9. Any straight line is drawn in a given direction to cut two fixed hyperbolas which have the same asymptotes in the points  $P$ ,  $P'$  and  $Q$ ,  $Q'$  respectively. Shew that the rectangle  $PQ$ ,  $QP'$  is constant.

10.  $PN$  is the ordinate of any point  $P$  on a rectangular hyperbola, and  $NQ$  is a tangent to the auxiliary circle. Shew that  $PQ$  passes through a vertex of the hyperbola.

11. Parallel tangents are drawn to a system of circles which pass through two given points; shew that the locus of the points of contact of the tangents is a rectangular hyperbola.

12. Pairs of equal circles are drawn through the points  $A$ ,  $B$  and  $A$ ,  $C$  respectively. Shew that their other point of intersection is on a rectangular hyperbola through  $A$ ,  $B$ ,  $C$  and having  $BC$  for a diameter.

13. A point moves so that the lines joining it to two fixed points make equal angles with a fixed straight line. Shew that the locus of the point is a hyperbola.

14. Shew that if an equilateral triangle be inscribed in a rectangular hyperbola, the centre of its circum-circle will be on the curve.

15. Shew that the locus of the points of intersection of two equal circles which touch two given parallel straight lines at fixed points  $A, A'$  respectively, and whose centres are on the same side of  $AA'$ , is a rectangular hyperbola.

16. If any parallelogram be inscribed in a rectangular hyperbola, the rectangle contained by the two perpendiculars drawn from any point of the curve to one pair of parallel sides is equal to the rectangle contained by the perpendiculars drawn from the same point on the other pair of parallel sides.

17. Points  $P, Q$  are taken on a rectangular hyperbola and the conjugate hyperbola respectively such that  $PQ$  subtends a right angle at the common centre. Shew that the locus of the middle point of  $PQ$  is another rectangular hyperbola whose asymptotes are the axes of the original curves.

18. Tangents are drawn to a hyperbola at the points where it is met by any tangent to the conjugate hyperbola; shew that the points of intersection of these tangents will be on the conjugate hyperbola.

19. Prove that the common chords of a hyperbola and any circle may be grouped in pairs which meet the asymptotes in concyclic points, and that these circles are concentric with the original circle.

20. The normals at the points  $Q, Q'$  of a conic intersect at right angles in the point  $O$  and cut the curve again in the point  $q, q'$  respectively. Shew that  $qq'$  is parallel to  $QQ'$ .

21. A straight line cuts the asymptotes of a conic in  $R, R'$  and any pair of conjugate diameters in  $P, P'$ ; shew that, if  $V$  be the middle point of  $RR'$ ,  $VR^2 = VP \cdot VP'$ .

22. With one focus of a given hyperbola as focus and any tangent to the hyperbola as directrix is described another hyperbola touching the conjugate axis of the former; prove that the two hyperbolas will be similar.

23. If from any point on a hyperbola a tangent be drawn to its auxiliary circle, the tangent will be equal to the semi-minor axis of the confocal ellipse through the point.

24. If the tangents at the ends of a chord of a hyperbola meet in  $T$ , and  $TM, TM'$  be drawn parallel to the asymptotes to meet them in  $M, M'$ ; then will  $MM'$  be parallel to the chord.

25. A circle cuts a rectangular hyperbola in the points  $P, P', Q, Q'$ . Shew that the tangents at  $P, P'$  will meet on the diameter perpendicular to  $QQ'$ .

26. Through a given point  $P$  any straight line is drawn meeting two fixed straight lines in the points  $Q, Q'$  respectively, and the point  $P'$  is taken on the line such that  $QP = P'Q'$ . Shew that the locus of  $P'$  is a hyperbola.

27. If a line pass through a fixed point, the locus of the middle point of the portion intercepted between two given straight lines is a hyperbola.

28.  $TQ, TQ'$  are tangents to a rectangular hyperbola whose centre is  $C$ ; shew that, if the bisectors of the angle  $QTQ'$  meet  $QQ'$  in  $K, K'$ , then will  $CK, CK'$  be the bisectors of the angle  $QCCQ'$ .

29. Through a fixed point  $O$  a chord  $PQ$  of a hyperbola is drawn, and lines  $PL, QL$  are drawn parallel to the asymptotes; shew that the locus of  $L$  is a hyperbola whose asymptotes are parallel to those of the given hyperbola and whose centre is at the fixed point  $O$ .

30. If  $P$  be any point on a hyperbola whose foci are  $S$  and  $H$ , and if the tangent at  $P$  meet an asymptote in  $T$ , the angle between that asymptote and  $HP$  will be double the angle  $STP$ .

31. Two tangents to a hyperbola from a point  $O$  cut a directrix in the points  $T, T'$ , and  $S$  is the focus corresponding to that directrix. Shew that the circle whose centre is  $O$  and which touches  $ST, ST'$  will cut the directrix in two points the radii to which from the point  $O$  are parallel to the asymptotes.

32. Describe a hyperbola through the angular points of a given parallelogram and having one asymptote in a given direction.

33. Prove that, if  $A, B$  and  $C$  be three fixed points, two parabolas can be drawn through  $A$  and  $B$  with  $C$  as focus, and that the axes of these parabolas are parallel to the asymptotes of the hyperbola which can be drawn through  $C$  with its foci at  $A$  and  $B$ .

34. A circle is described passing through  $P, P'$  the extremities of any diameter of a rectangular hyperbola, and cutting the tangent at  $P$  in  $T$ : prove that  $P'T$  and the tangent to the circle at  $P$  meet on the hyperbola.

35. The tangent at a fixed point of a rectangular hyperbola meets  $CT, CT'$ , any pair of conjugate diameters, in  $T, T'$ . Shew that the locus of the centre of the circle  $CTT'$  is a straight line.

36.  $PP'$  is any diameter of a rectangular hyperbola and  $Q$  is any point on the curve.  $PR, P'R'$  are drawn at right angles to  $PQ, P'Q'$  respectively, intersecting the normal at  $Q$  in  $R, R'$ . Prove that  $QR$  and  $Q'R'$  are equal.

37. The locus of the foci of conics which touch the four sides of a parallelogram is a rectangular hyperbola.

38. A circle and a rectangular hyperbola circumscribe a right-angled triangle  $ABC$ ,  $C$  being the right angle, and the tangent to the circle at  $C$  meets the hyperbola again in  $C'$ ; prove that the tangents to the hyperbola at  $C$  and  $C'$  intersect  $AB$ .

39. Shew that, if the tangents from a point to a given conic make equal angles with a given straight line, the point must lie on a rectangular hyperbola through the foci of the conic.

40. Find the centre of a rectangular hyperbola having given three points on the curve and the tangent at one of them.

41.  $OT$  is the tangent at  $O$  to a rectangular hyperbola,  $PQ$  a chord meeting the tangent at right angles at  $T$ ; shew that the two bisectors of the angle  $OCT$  bisect the lines  $OP$  and  $OQ$ .

42. The locus of the points of intersection of tangents to an ellipse which make equal angles with the major and minor axes respectively but are not at right angles, is a rectangular hyperbola whose vertices are the foci of the ellipse.

43. The locus of the extremities of parallel diameters of a system of co-axial circles is a rectangular hyperbola.

44. A square is circumscribed to a circle and any tangent to the circle meets two parallel sides of the square in the points  $P, Q$ , and a parallel tangent to the circle meets the other two sides of the square in the points  $R, S$ . Shew that the points  $P, Q, R, S$  lie on a rectangular hyperbola passing through the centre of the circle and whose centre is on the circle.

45. A chord  $PP'$  of a hyperbola cuts the asymptotes in  $R, R'$ ;  $CTV$  is the diameter bisecting the chord in  $V$ , and  $T$  is the intersection of the tangents at the extremities of the chord. Prove that the parallelogram described with  $TV$  as diagonal and its sides parallel to the asymptotes, has its other corners on the curve and its other diagonal parallel to  $PP'$  and a third proportional to  $RV$  and  $PV$ .

46. Shew that, if points be taken on a fixed diameter of a central conic, and perpendiculars be drawn from them on their polars, the locus of the feet of these perpendiculars is a rectangular hyperbola.

47.  $PP'$  is a diameter of a rectangular hyperbola, and a circle with centre  $P$  and radius  $PP'$  cuts the hyperbola again in the points  $A, B, C$ . Shew that  $ABC$  is an equilateral triangle.

48. Construct a hyperbola, having given three points on the curve and the directions of the asymptotes.

49. Through a focus  $S$  and the further vertex  $A'$  of a hyperbola whose eccentricity is 2, a circle is drawn cutting the hyperbola in the points  $A', P, Q, R$ . Shew that the triangle  $PQR$  is equilateral.

50. If an asymptote and two points of a conic be given, the axes will envelop a parabola.

51. Shew that, if a conic touch the sides  $BC, CA, AB$  of a triangle  $ABC$  in  $P, Q, R$  respectively, the lines  $AP, BQ, CR$  will meet in a point.

52. A conic touches the sides  $BC, CA, AB$  of a triangle in the points  $P, Q, R$  respectively, and the lines  $QR, RP, PQ$  cut  $BC, CA, AB$  respectively in  $L, M, N$ ; shew that  $L, M, N$  are on a straight line.

53. A conic cuts the sides  $BC$ ,  $CA$ ,  $AB$  of the triangle  $ABC$  in the points  $P$ ,  $P'$ ;  $Q$ ,  $Q'$  and  $R$ ,  $R'$  respectively. Shew that

$$BP \cdot BP' \cdot CQ \cdot CQ' \cdot AR \cdot AR' = BR \cdot BR' \cdot AQ \cdot AQ' \cdot CP \cdot CP'.$$

[Carnot's Theorem.]

54. From any point on one hyperbola tangents are drawn to another having the same asymptotes; shew that the chord of contact cuts off a constant area from the asymptotes.

55. A circle intersects a hyperbola in four points; prove that the product of the distances of the four points of intersection from one asymptote is equal to the product of their distances from the other.

56. Shew that, if a rectangular hyperbola cut a circle in four points the centre of mean position of the four points is midway between the centres of the two curves.

57. Having given five points on a circle; shew that the centres of the five rectangular hyperbolas, each of which passes through four of the given points, will all lie on a circle whose radius is half that of the given circle.

58. From an external point two tangents are drawn to a given conic; shew that, if the four points where the tangents cut the axes of the conic lie on a circle, the point from which the tangents are drawn must lie on a fixed rectangular hyperbola through the foci of the conic.

59. If the tangents at the points  $Q$ ,  $Q'$  on a given conic be at right angles to one another, the line  $QQ'$  will always touch a fixed confocal conic.

60. From any point  $T$ , the tangents  $TP$ ,  $TP'$  are drawn to one conic and  $TQ$ ,  $TQ'$  to a confocal conic. Shew that  $PQ$ ,  $PQ'$  are equally inclined to the tangent at  $P$ .

61.  $PP'$ ,  $QQ'$  are the chords of contact of pairs of tangents from a point  $T$  to each of two confocal conics, whose foci are  $S$ ,  $S'$ . Prove that, if  $P$ ,  $Q$ ,  $S$  are collinear,  $QS'P'$  and  $PSQ'$  will also be collinear; prove also that the locus of  $T$  will be a straight line perpendicular to  $SS'$ .

62.  $TP$  is a tangent to one conic, and  $TQ$  is a perpendicular tangent to a confocal conic. Shew that the line joining  $T$  to the centre of the conics will bisect  $PQ$ .

63.  $TP$  is a tangent to a fixed conic and  $TQ$  is a perpendicular tangent to a fixed confocal conic. Shew that  $PQ$  touches a third fixed confocal.

64. Tangents parallel to a given straight line are drawn to a system of confocal conics; shew that their points of contact lie on a rectangular hyperbola through the foci.

65. A parallelogram circumscribes a conic and has its sides parallel to two fixed straight lines; shew that the four angular points are on a fixed rectangular hyperbola for all conics of a given confocal system.

66. Shew that, if  $P$  be any point on an ellipse and  $p$  be the corresponding point on its auxiliary circle, an asymptote of the confocal hyperbola through  $P$  will pass through  $p$ .

67. Tangents are drawn to a given system of confocal conics from any fixed point on the transverse axis. Shew that their points of contact lie on a circle.

68. If the sides of a triangle which is inscribed in one conic touch a confocal conic, the points of contact will lie on the escribed circles of the triangle.

69. Shew that, if the line through a point perpendicular to its polar with respect to a given conic pass through a fixed point  $O$ , the polar will envelope a parabola which touches the axes of the given conic. Prove also that the same parabola is obtained if the conic be any one of a given confocal system.

70. Shew that, if  $OP$ ,  $OQ$  be tangents to a conic, the normals at  $P$ ,  $Q$  and the line  $PQ$  will all touch a parabola which touches the axes of the conic.

71.  $ABC$  is a triangle inscribed in an ellipse, and a confocal ellipse touches the sides in  $A'$ ,  $B'$ ,  $C'$  respectively. Prove that the points  $A$ ,  $A'$  are on the same confocal hyperbola.

72. A line through one focus is drawn to meet a system of confocal conics: prove that the tangents to the conics at the points of intersection all touch a fixed parabola.

73.  $TQ$ ,  $TQ'$  are perpendicular tangents to an ellipse, and  $TR$ ,  $TR'$  are tangents to an interior confocal. Shew that,  $R$ ,  $R'$ ,  $O$ ,  $O'$  are cyclic, where  $O$ ,  $O'$  are the intersections of  $QR'$ ,  $Q'R$  and  $QR$ ,  $Q'R'$  respectively.

74. Shew that, if tangents  $TQ$ ,  $TQ'$  be drawn from a fixed point  $T$  to any one of a system of confocal conics, the circle  $TQQ'$  will pass through another fixed point.

75.  $Q$ ,  $Q'$  are any two points on an ellipse whose foci are  $S$ ,  $S'$ ;  $QS$ ,  $Q'S'$  intersect in  $M$  and  $QS'$ ,  $Q'S$  intersect in  $N$ , and the tangents at  $Q$ ,  $Q'$  intersect in  $T$ . Shew that  $M$  and  $N$  are on a confocal hyperbola and that  $TN$ ,  $TM$  are tangents to that hyperbola.

## CHAPTER V.

### SECTIONS OF A CONE.

**126. Def.** The surface generated by a straight line which passes through a fixed point  $V$ , and which moves so as always to intersect the circumference of a circle whose plane is perpendicular to the line joining its centre,  $C$ , to the point  $V$ , is called a **right circular cone**, of which the point  $V$  is called the **vertex**, and the line  $CV$  is called the **axis**.

**127.** *Every plane section of a right circular cone is a conic.*

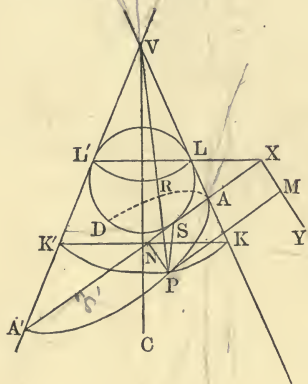
It is easily seen that every section perpendicular to the axis of the cone is a circle.

Let  $DAP$  be any plane section, and let the plane of the paper contain the axis  $VC$  of the cone, and be perpendicular to the plane  $DAP$ ,  $VK$  and  $VK'$  being the generating lines in the plane of the paper.

Let the cutting plane intersect the plane  $KVK'$  in the line  $ASN$ .

Then a sphere can be described (whose centre is that of the circle which touches the three lines  $VK$ ,  $VK'$  and  $AN$ ) which will touch the plane  $DAP$  in some point  $S$  on  $AN$ , and will touch the cone along a circle  $LRL'$  suppose,  $LL'$  being the diameter of the circle in the plane of the paper.

Let  $XY$  be the line of intersection of the cutting plane and the plane of contact of the cone and sphere; then, since these planes are both perpendicular to the plane of the paper,  $XY$  is perpendicular to the plane of the paper and therefore perpendicular to  $ASN$ .



Through any point  $P$  on the curve  $DAP$  draw a plane perpendicular to the axis of the cone, and let this plane cut  $ASN$  in the point  $N$  and the cone in the circle  $KPK'$ . Then  $PN$  will be perpendicular to  $ASN$  and therefore parallel to  $XY$ .

Join  $PS$ , and draw  $PM$  perpendicular to  $XY$ .

Then, if  $PV$  cut the circle  $LRL'$  in the point  $R$ ,  $PS$  and  $PR$  are both tangents to the sphere and are therefore equal. Also  $PMXN$  is a rectangle, and  $PM = NX$ .

Hence

$$\begin{aligned} SP : PM &= PR : NX \\ &= KL : NX \\ &= AL : AX \\ &= AS : AX. \end{aligned}$$

Hence the curve  $DAP$  is a conic whose focus is  $S$  and directrix  $XY$ .

The conic is an ellipse, parabola, or hyperbola according as  $AS$  or  $AL$  is less, equal, or greater than  $AX$ , that is according as the angle  $LXA$  is less, equal, or greater than the angle  $ALX$  or  $LL'V$ .

Thus the curve is an ellipse or hyperbola according as the cutting plane intersects  $KA$ ,  $K'A$  in two points on the same side or on opposite sides of the vertex  $V$ , and the curve is a parabola when the cutting plane is parallel to one of the generating lines of the cone.

**Cor. I.** *The sections of a given right circular cone by parallel planes are conics whose eccentricities are equal.*

**Cor. II.** *The angle between the asymptotes of a hyperbolic section of a right circular cone is equal to the angle between the two straight lines in which the cone is cut by a parallel plane through its vertex.*

**128.** Another proof that a plane section of a right circular cone is a conic can be given, provided the section is not parallel to one of the generators of the cone.

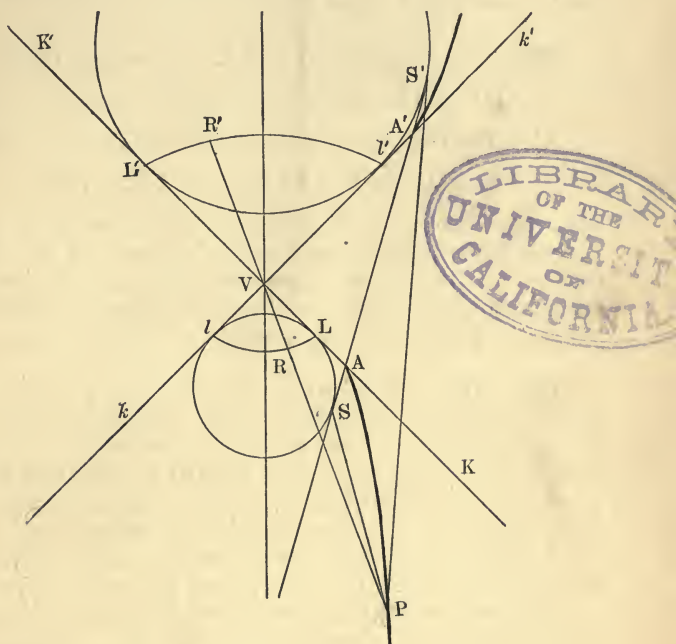
For, let the plane of the paper contain the axis of the cone and be perpendicular to the cutting plane, and let  $KVK'$  and  $kvk'$  be the generating lines of the cone in the plane of the paper.

Then, the line of intersection of the cutting plane and the plane  $KVk$  is not, by supposition, parallel to either of the generators  $KK'$  or  $kk'$ ; it will therefore meet them in points  $A$ ,  $A'$  respectively.

Now two spheres can be described each touching the cutting plane in some point in the line  $AA'$  and touching the cone along a circle whose plane is perpendicular to the axis of the cone. [The centres of these spheres will be centres of circles in the plane  $KVk$  which touch the three lines  $AV$ ,  $A'V$  and  $AA'$ .]

Let  $S$ ,  $S'$  be the points in which these spheres touch the cutting plane, and let  $LRL$ ,  $L'R'l'$  be the circles along which they touch the cone.

Let  $P$  be any point on the section. Join  $PS$ ,  $PS'$  and  $PV$ , and let  $PV$  cut the circles of contact of the spheres in the points  $R$ ,  $R'$  respectively.



Then  $PS = PR$ , since they touch the same sphere.

Similarly  $PS' = PR'$ .

Hence, if  $A$  and  $A'$  are on the same side of the vertex

$$PS + PS' = RR'.$$

And, if  $A$ ,  $A'$  are on opposite sides of the vertex (as in the figure)

$$PS - PS' = RR'.$$

But  $VR$  and  $VR'$  are obviously constant, and therefore  $RR'$  is constant.

Thus the plane section is a conic whose foci are the points of contact of the two spheres which can be inscribed in the cone so as to touch the plane of the section.

129. From the figure to Art. 127, we see that

$$A'V - AV = A'L' - AL = A'S - AS = SS'.$$

Also, from the figure to Art. 128, we see that

$$A'V + AV = A'l + AL = A'S + AS.$$

Hence the vertex of a right circular cone which passes through a given ellipse (or hyperbola) must lie on a hyperbola (or an ellipse) in a perpendicular plane through its transverse axis whose foci are the vertices and whose vertices are the foci of the ellipse (or hyperbola).

130. **Def.** The surface generated by a straight line which is always perpendicular to the plane of a given circle, and moves so as always to intersect the circumference of the circle, is called a **right circular cylinder**, of which the straight line through the centre of the circle perpendicular to its plane is called the **axis**.

A right circular cylinder is therefore the limiting form of a right circular cone as its vertex passes off to infinity.

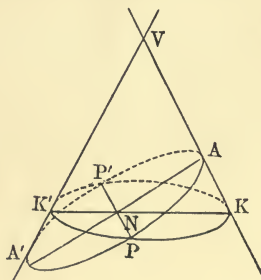
It is obvious that all sections of the cylinder by planes perpendicular to its axis are equal circles; also that any section by a plane parallel to the axis is a pair of parallel straight lines, which become coincident when the plane touches the cylinder.

It can be proved by the method of Art. 128 that every other section of a right circular cylinder is an ellipse, whose foci are the points of contact of the two spheres which are inscribed in the cylinder so as to touch the plane of the section.

## EXAMPLES.

1. Find the locus of the foci of parallel sections of a given right circular cone.
2. Find the least angle of a cone which it is possible to cut by a plane so that the section is a rectangular hyperbola.
3. The minor axis of any elliptic section of a cone is a mean proportional between the diameters of the circular sections of the cone which pass through the extremities of its major axis.
4. The minor axes of all elliptic sections of a right circular cylinder are equal.

**131.** The method adopted in Art. 127 gives a focus and the corresponding directrix of any plane section of a right circular cone. We may, however, prove that a plane section is a conic without finding a focus or a directrix. Let any cutting plane cut the perpendicular plane through the axis of the cone in the line  $AA'$ , the points  $A, A'$  being on the same side of the vertex,  $V$ , of the cone.



Let  $P$  be any point on the curve, and draw through  $P$  a plane perpendicular to the axis of the cone, and cutting  $AA'$  in  $N$ .

Then, this plane will cut the cone in a circle of which  $KK'$  is a diameter,  $K$  and  $K'$  being the points in which the plane cuts the generating lines  $VA, VA'$  respectively. Also  $KK'$  will pass through  $N$  and will be perpendicular to  $PN$ .

Hence  $PN^2 = KN \cdot NK'$ .

Now each of the triangles  $KNA, K'NA'$  has its sides in fixed directions, for all positions of  $N$ . Hence, for all positions of  $P$ , the ratios  $KN : AN$  and  $NK' : NA'$ , and therefore also

$$KN \cdot NK' : AN \cdot NA',$$

are constant.

Hence the ratio  $PN^2 : AN \cdot NA'$  is constant for all points on the curve, whence it follows that the curve is an *ellipse*.

If  $A, A'$  be on opposite sides of the vertex, it can be proved in a similar manner that the plane section is a hyperbola; and also that when the cutting plane is parallel to a generating line of the cone, the section is a parabola.

**132.** The following theorem is a generalisation of the focus and directrix definition of a conic:—

*If a circle touch a conic at  $P$  and  $P'$ , the extremities of a double ordinate  $PNP'$  to the transverse axis, the tangent from any point  $Q$  of the conic to this circle is to the perpendicular distance of  $Q$  from  $PP'$  in the ratio of the eccentricity.*

Let  $V$  be the vertex of a right circular cone which passes through the conic  $APA'$ ,  $AA'$  being the transverse axis of the conic. Draw a circular section  $LPL'P'$  of the cone through the points  $P, P'$ ;  $L, L'$  being on the generating lines  $VA, VA'$  respectively. Then a sphere can be described which will touch the cone at all points on the circle  $LPLP'$ . The section of this sphere by the plane  $APA'P'$  will therefore be a circle touching the conic at the two points  $P, P'$ .

Let  $Q$  be any point on the conic, and let the circular section of the cone through  $Q$  cut  $AA'$  in  $M$ , and  $VA, VA'$  in  $K, K'$  respectively.

Let  $QV$  cut the circle  $LPL'P'$  in  $R$ .

Then the tangent from  $Q$  to the circle in which the sphere is cut by the plane  $APA'P'$  will be a tangent to the sphere, and will therefore be equal to  $QR$ ; also the perpendicular distance of  $Q$  from  $PP'$  will be equal to  $MN$ .

But

$$QR : MN = KL : MN$$

$$= LA : NA$$

$$= AS : AX,$$

which proves the proposition.

**Cor.** *The sum or difference of the tangents drawn from any point of a conic to two circles each of which touches the conic at the extremities of a chord perpendicular to its transverse axis is constant.*

## EXAMPLES ON CHAPTER V.

1. Shew how to cut a given cone so that the section may be a parabola of given latus rectum.
2. Shew that, if the vertical angle of a cone be a right angle, the major axis of any elliptic section is equal to the difference of the radii of the focal spheres.
3.  $P, P'$  are the extremities of any diameter of a given elliptic or hyperbolic section of a right circular cone; shew that the sum of the distances of  $P$  and  $P'$  from the vertex of the cone is constant.
4. Shew that, if two sections of a right circular cone have a common directrix, the latera recta of the sections are in the ratio of their eccentricities.
5. Shew that the minor axis of an elliptic section of a right circular cone is a mean proportional to the diameter of the focal spheres.
6. Shew that the latus rectum of any plane section of a given right circular cone varies as the perpendicular from the vertex of the cone on the plane of section.
7. If two different sections of a cone have a common directrix, the line joining their foci will pass through the vertex of the cone.
8. Shew that two elliptic sections of a given cone can be found which have a given point within the cone for focus.
9. Shew that, if two cones be described so as to touch two given spheres, the ratio of the eccentricity of the two conics in which they are cut by any plane will be constant.
10. Prove that the axis of every right circular cone which has a given central conic for one of its plane sections will touch a central conic, which is an ellipse or a hyperbola according as the given conic is a hyperbola or an ellipse.
11. Elliptic sections of a right circular cone are made by planes perpendicular to a given plane containing the axis of the cone, and the ellipses have their minor axes of constant length. Shew that the locus of their centres is a hyperbola.
12. Two cones which have a common vertex, their axes at right angles, and their vertical angles supplementary, are intersected by a plane at right angles to the plane of their axes. Prove that the distances of either focus of the elliptic section from the foci of the hyperbolic section are equal to the distances of the vertices of the ellipse from the vertex of the cone.
13. Shew that the locus of the centres of plane sections of a given right circular cone, drawn through a given point on the axis of the cone, is the surface formed by the revolution of a hyperbola about its transverse axis.

14. Shew that if the latus rectum of a plane section of a given right circular cone be of given length, the foci lie on the surface generated by the revolution of a hyperbola about its transverse axis.

15. If  $O, O'$  be the centres of the two spheres inscribed in a right circular cone so as to touch any plane, the sphere on  $OO'$  as diameter will pass the auxiliary circle of the section of the cone made by that plane.

16. Shew that the director-circle of any plane section of a cone lies on the sphere which passes through the two circles of contact of the focal spheres.

17. Prove that a right cylinder on a given elliptic base can be cut in two ways so that the curve of section may be a circle; and that a sphere can always be drawn through any two circular sections which are not parallel.

18. A surface is formed by the revolution of an ellipse about its major axis, and a plane is drawn cutting the surface and touching at a point  $S$  a sphere inscribed in the surface. Shew that the section is an ellipse of which  $S$  is a focus.

19. Shew that the centres of elliptic sections of a right circular cone which have major axes of equal length lie on the surfaces generated by the revolution of an ellipse about one of its axes.

20. Shew that the locus of the vertices of the right circular cones which pass through one conic is a conic of the other species in a perpendicular plane, whose vertices are the foci and whose foci are the vertices of the former; and deduce that the sum or difference of the distances of a variable point on one of these conics from any two fixed points on the other is constant.

## CHAPTER VI.

### ORTHOGONAL PROJECTION. SIMILARITY OF CURVES. CURVATURE OF CONICS.

**133. Def.** The foot of the perpendicular from a point on a fixed plane is called the **orthogonal projection** of the point on that plane, and the fixed plane is called the **plane of projection**.

If a point describe any curve its orthogonal projection on a given plane will describe a curve which is called the orthogonal projection of the given curve.

In general, if any point  $P$  be joined to a fixed point  $V$ , and  $VP$  be cut by any fixed plane in the point  $P'$ , the point  $P'$  is called the projection of  $P$  on the fixed plane; also the point  $V$  is called the *centre of projection*, and the fixed plane is called the *plane of projection*.

Thus orthogonal projection is only a particular case, when the centre of projection is at an infinite distance and in a direction perpendicular to the plane of projection.

**134.** The principal properties of orthogonal projection are the following:—

(i) *The projection of a straight line is a straight line.*

For, let the given line cut the plane of projection in the point  $A$ , and let  $P'$  be the projection of any point  $P$  on the given line. Then, if  $Q$  be any other point on the given line, and if  $QQ'$  be the perpendicular from  $Q$  on the line  $AP'$ , then  $QQ'$  will be parallel to  $PP'$  and therefore also perpendicular to the plane of projection, so that  $Q'$  is the projection of  $Q$ . Thus the projection of every point on  $AP$  will lie on the line  $AP'$ .

(ii) *Parallel straight lines project into parallel straight lines.*

For the projection of the point of intersection of two straight lines is the point of intersection of their projections, and if one of these points is at infinity the other must also be at infinity. Hence, if the original lines are parallel, the projected lines will be parallel; and, conversely, if the projections are parallel the original lines must have been parallel.

(iii) *Parts of the same straight line, or of parallel straight lines, are in the same ratio as their projections.*

For let  $A'B'$  and  $C'D'$  be respectively the projections of the parallel straight lines  $AB$  and  $CD$ .

Draw lines through  $A$  and  $C$  parallel to  $A'B'$  or  $C'D'$  and meeting  $BB'$ ,  $CC'$  respectively in the points  $X$ ,  $Y$ .

Then the triangles  $XAB$ ,  $YCD$  will be similar, and therefore

$$\begin{aligned} AB : AX &= CD : CY; \\ \therefore AB : CD &= AX : CY \\ &= A'B' : C'D', \end{aligned}$$

since  $AX = A'B'$  and  $CY = C'D'$ .

(iv) *The number of points in which a curve is cut by a straight line (or one plane curve is cut by another) is equal to the number of points of intersection of their projections.*

(v) *The projection of a tangent to a curve is a tangent to the projection of the curve.*

For, if two points of intersection of a straight line and a curve be coincident, two points of intersection of their projections will also be coincident.

Conversely, if the projections of a straight line and a curve touch one another, the straight line and the curve must themselves touch.

(vi) *The area of any curve on a given plane and of its projection on another given plane are in a constant ratio.*

Divide the given area into any number of rectangles by two sets of equidistant lines parallel and perpendicular respectively to the line of intersection of the given plane and the plane of projection. Then, those segments which are parallel to the line of intersection will be unaltered by projection, and those which are perpendicular will be diminished in a constant ratio. [This ratio will be  $1 : \cos \theta$ ,

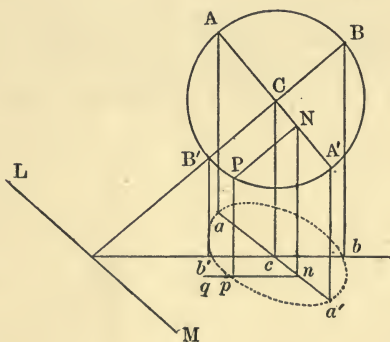
where  $\theta$  is the angle between the planes.] Hence every rectangle, and therefore the sum of any number of rectangles, will be diminished by projection in a constant ratio. But when the parallel lines are drawn indefinitely near together, so that each of the rectangles is made indefinitely small, their sum is ultimately equal to that of the area in which they are drawn. Hence any area in a given plane is in a constant ratio to the area of its projection on any other given plane.

**135.** *The projection of a circle is an ellipse.*

Let  $LM$  be the line of intersection of the plane of the circle and the plane of projection.

Let  $ACA'$  be the diameter of the circle which is parallel to  $LM$ , and let  $BCB'$  be the perpendicular diameter.

Let  $aca'$  and  $bcb'$  be the projections of  $ACA'$  and  $BCB'$  respectively; then since  $ACA'$  is parallel to the plane of projection,  $aca' = ACA'$ , and  $bcb'$  is perpendicular to  $aca'$ .



Let  $NP$  be any ordinate to the diameter  $ACA'$  of the circle, and let  $np$  be its projection. Also, let  $np$  cut the circle on  $aa'$  as diameter in the point  $q$ .

Then, since the circles  $ABA'$  and  $aq a'$  are equal and  $cn = CN$ ,  $nq$  must be equal to  $NP$ .

Now  $np$  and  $cb$  are the projections of the *parallel* straight lines  $NP$  and  $CB$  respectively ;

$$\therefore np : cb = NP : BC$$

$$= nq : ca.$$

Hence

$$np : nq = cb : ca,$$

whence it follows that *the locus of  $p$  is an ellipse of which the circle  $aca'$  is the auxiliary circle.*

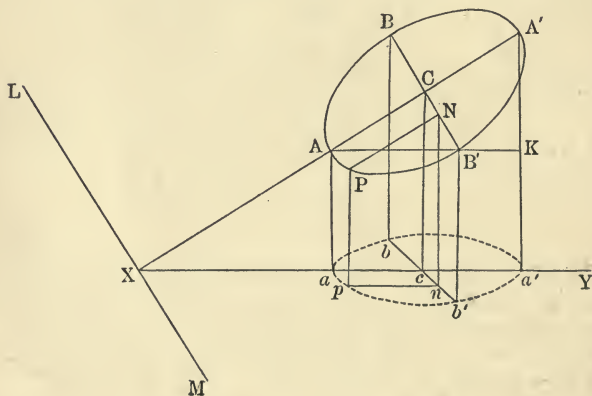
**136.** *An ellipse can be projected into a circle.*

Let  $ACA'$ ,  $BCB'$  be the major and minor axes of an ellipse.

Draw a plane through  $AA'$  perpendicular to  $BCB'$ , and in this plane describe a circle on  $AA'$  as diameter, and let the chord  $AK$  of the circle be equal to  $BCB'$ .

In the plane of the ellipse draw any line  $LM$  parallel to  $BCB'$  and cutting the major axis in the point  $X$ . Then, if  $XY$  be parallel to  $AK$ , the projection of the ellipse on the plane  $LMY$  will be a circle equal to the minor auxiliary circle of the given ellipse.

For, since  $XY$  is parallel to  $AK$ ,  $XY$  must lie in the



plane  $A'AK$ . And, since  $BCB'$  is perpendicular to the plane  $A'XY$ , so also is the parallel line  $LXM$ . Hence  $a, a'$  the projections of  $A, A'$  respectively on the plane  $LMY$ , will lie on the line  $XY$ .

Let  $b, b'$  be the projections of  $B, B'$  respectively; then, since  $BCB'$  is parallel to the plane of projection,

$$bcb' = BCB';$$

also  $bcb'$  is perpendicular to  $aca'$ .

Let  $NP$  be any ordinate to the diameter  $BCB'$  of the ellipse, and let  $np$  be the projection of  $NP$ .

Then, since  $NP$  is parallel to  $CA$ ,  $np$  is parallel to  $ca$ , and

$$\begin{aligned} np : ca &= NP : CA \\ &= NQ : BC, \end{aligned}$$

if  $Q$  be the point where  $NP$  cuts the minor auxiliary circle of the ellipse.

Hence, as  $ca$  is by construction equal to  $BC$ ,  $np = NQ$ . But  $cn = CN$ , and  $np$  is perpendicular to  $cn$ , whence it follows that the locus of  $p$  is a circle equal to the minor auxiliary circle of the ellipse.

**137.** If a central conic be orthogonally projected into any other conic; then, since every chord through the centre of the original conic is bisected in that point, and the segments of a straight line are in the same ratio as their projections, it follows that *the projection of the centre of the original conic will be the centre of the projection.*

Also, since tangents project into tangents and parallel lines into parallel lines, it follows that *any pair of conjugate diameters of the original conic will project into a pair of conjugate diameters of the projection.*

**Ex. 1.** Shew that every orthogonal projection of a conic is a conic of the same species whose centre is the projection of the centre of the original conic. [Use Art. 45, Art. 81 or Art. 104.]

**Ex. 2.** Shew that any two intersecting straight lines can be orthogonally projected into perpendicular straight lines.

Ex. 3. Shew that any hyperbola can be orthogonally projected into a rectangular hyperbola.

Ex. 4. Shew that the ratio of the area of an ellipse to the area of its auxiliary circle is equal to the ratio of the minor to the major axis.

**138.** Many of the properties of an ellipse may be proved by projecting the ellipse into a circle. These are called *projective properties*.

Ex. 1. *The locus of the middle points of parallel chords of an ellipse is a straight line.*

For project the ellipse into a circle.

Then the system of parallel chords will project into a system of parallel chords, and the middle points of the original chords will project into the middle points of the projected chords.

Thus we have to prove that the locus of the middle points of parallel chords of a circle is a straight line, which follows at once from Euclid III. 3.

Ex. 2. *If the tangents at the extremities of the chord  $QQ'$  of an ellipse whose centre is  $C$  meet in the point  $T$ , and  $CT$  cut  $QQ'$  in  $V$  and the ellipse in  $P$ , then will  $CV \cdot CT = CP^2$ .*

Project the ellipse into a circle; then the centre of the ellipse will project into the centre of the circle, for every chord of the circle through the projection of the centre of the ellipse will be bisected at that point.

Let  $c, t, q, q', v$  be the projections of  $C, T, Q, Q', V$  respectively; then since a tangent to any curve will project into a tangent to the projection of the curve,  $tq$  and  $tq'$  will touch the circle.

Also, since parts of the same straight line are in the same ratio as their projections;

$$CV : CP = cv : cp,$$

and  $CP : CT = cp : ct.$

But, since the triangles  $cvq$  and  $cqt$  are similar,

$$cv : cq = cq : ct;$$

$$\therefore cv : cp = cp : ct.$$

Hence  $CV : CP = CP : CT.$

## SIMILARITY OF CURVES.

**139. Def.** Two curves are said to be **similar and similarly situated** when radii drawn to the first from a certain fixed point  $O$  are in a constant ratio to parallel radii drawn to the second from another fixed point  $O'$ .

Two curves are **similar**, but not similarly situated, when radii making a constant angle with one another, drawn to the curves from two fixed points  $O, O'$  respectively, are proportional.

The two fixed points  $O, O'$  are called **centres of similarity**; and when  $O'$  coincides with  $O$  the point is called a **centre of similitude** of the two curves.

**140.** *If one pair of centres of similarity exist for two curves, there will be an infinite number of such pairs, and if the curves be similarly situated a centre of similitude can be found.*

Let  $O, O'$  be the given centres of similarity, and let  $OP, O'P'$  be any pair of corresponding radii. Take  $C$  any point whatever, and draw  $O'C'$  making the same angle with  $O'P'$  that  $OC$  makes with  $OP$ , and let  $C'$  be such that the ratio  $O'C' : OC$  is equal to the constant ratio  $O'P' : OP$ . Then the triangles  $COP, C'O'P'$  will be similar and therefore  $C'P'$  will make the same angle with  $CP$  that  $O'P'$  makes with  $OP$ , moreover  $C'P' : CP$  will be equal to the constant ratio  $O'P' : OP$ . Hence  $C$  and  $C'$  are centres of similarity for the two curves.

Again, if the curves be similarly situated and  $OP, O'P'$  be any pair of parallel radii; then, if  $PP'$  cut  $OO'$  in the point  $A$ ,  $OA : O'A = OP : O'P' = \text{constant}$ . Hence  $A$  is a fixed point for all directions of the parallel radii; also  $AP : AP' = OP : O'P' = \text{constant}$ . Hence  $A$  is a centre of similitude.

**141.** *If two curves be similar the tangents at corresponding points will be inclined at a constant angle to one another.*

Let  $O, O'$  be centres of similarity of the two curves, and let  $OP, O'P'$  and  $OQ, O'Q'$  be any two pairs of corresponding radii.

Then the angles  $POQ, P'O'Q'$  will be equal and  $OP : O'P' = OQ : O'Q'$ .

Hence the triangles  $POQ, P'O'Q'$  are equiangular, and therefore the angle between  $PQ$  and  $P'Q'$  is equal to the constant angle between  $OP$  and  $O'P'$ . When  $OQ$  and  $O'Q'$  move up to coincidence with  $OP$  and  $O'P'$  respectively,  $PQ$  and  $P'Q'$  ultimately become the tangents at  $P$  and  $P'$  respectively.

**142.** *If two central conics be similar and similarly situated the centres of the two curves will be centres of similarity, and the conics must be of equal eccentricity.*

Let  $O$  and  $O'$  be two centres of similarity of the two curves. Draw any chord  $POQ$  of the one, and the parallel chord  $P'O'Q'$  of the other. Then by supposition

$$PO \cdot OQ : P'O' \cdot O'Q'$$

is constant for every pair of corresponding chords. But, since  $O$  is a fixed point,  $PO \cdot OQ$  is always in a constant ratio to the square of the parallel semi-diameter of the first conic; and the same applies to the other conic. Hence parallel diameters of the two conics are in a constant ratio to one another.

Since parallel diameters of the two conics are in a constant ratio to one another, the greatest or least diameter of one conic must be parallel to the greatest or least diameter of the other, and therefore the axes of the two conics must be parallel.

Also, if  $CP, C'P'$  be parallel diameters of the two curves, the tangents, and therefore the normals, at  $P, P'$  must be parallel; whence it follows that

$$CG : CN = C'G' : C'N',$$

which shews that the eccentricities of the two conics are equal.

The converse theorem, namely that conics of equal eccentricity are similar curves can be easily proved, the foci being centres of similarity.

**143.** *If two curves be similar and one of them is a conic the other must also be a conic.*

Let  $O, O'$  be the centres of similarity of the two curves, and let  $OP, O'P'$  be corresponding radii. Let  $S, H$  be the foci of the conic, and join  $SO, HO$ . Draw the lines  $O'S', O'H'$  such that the angles  $S'O'P', H'O'P'$  may be equal to the angles  $SOP, HOP$  respectively. Then, since  $OS$  and  $OH$  are fixed in direction,  $O'S'$  and  $O'H'$  are also fixed in direction. Hence, if the points  $S', H'$  be such that  $O'S' : OS = O'H' : OH = O'P' : OP = \text{constant}$ , the points  $S', H'$  will be fixed.

The triangles  $SOP, S'O'P'$  will be similar, and the triangles  $HOP, H'O'P'$  will be similar.

$$\text{Hence} \quad SP : S'P' = OP : O'P',$$

$$\text{and} \quad HP : H'P' = OP : O'P';$$

$$\therefore SP \pm HP : S'P' \pm H'P' \text{ is constant,}$$

whence it follows that, as  $P$  describes a conic whose foci are  $S$  and  $H$ ,  $P'$  will describe a conic whose foci are  $S'$  and  $H'$ .

The above holds good for all positions of the foci of the given conic, and therefore it will hold good in the limiting case when one of the foci is at an infinite distance. Thus the case when the conic is a parabola needs no separate investigation. A modified proof may, however, be given.

**Ex. 1.** *The middle points of all chords of a conic which pass through a fixed point  $O$  is a similar and similarly situated conic.*

Let  $C$  be the centre of the conic, and let  $CO$  cut the conic in the point  $A$ .

Let  $PP'$  be any chord of the conic through the fixed point  $O$  and let  $V$  be the middle point of  $PP'$ .

Draw the chord  $AA'$  of the conic parallel to  $PP'$  and let  $W$  be the middle point of  $AA'$ .

Then we know that  $CVW$  is a straight line.

Since  $AW = \frac{1}{2}AA'$ , the locus of  $W$  is a curve similar and similarly situated to the locus of  $A'$ ; the locus of  $W$  is therefore a conic similar and similarly situated to the given conic.

Again, since  $AW$  and  $OV$  are parallel,

$$CV : CW = CO : CA = \text{constant.}$$

Hence the locus of  $V$  is similar and similarly situated to the locus of  $W$ ; the locus of  $V$  is therefore a conic similar and similarly situated to the original conic.

**Ex. 2.** *The locus of the point of intersection of the tangents at the extremities of any two conjugate diameters of an ellipse is a similar and similarly situated ellipse.*

**Ex. 3.** *A chord of an ellipse passes through the ends of two conjugate diameters; shew that its middle point is on a similar and similarly situated ellipse.*

## CURVATURE.

**144.** The curvature of a curve at any point is the *rate of its bending* at that point. Now the curvature of a circle is obviously the same at all points, moreover the rate of bending of a circle changes continuously from infinity to zero as the radius of the circle changes from zero to infinity. Thus there is one and only one circle whose curvature is the same throughout as the curvature of a given circle at a given point.

If a circle be drawn through any three adjacent points  $P, Q, R$  of a curve, and if the points  $P, R$  move up to and ultimately coincide with the point  $Q$ ; then the curvature of the circle in its ultimate position will be equal to that of the curve at the point  $Q$ ; also this circle in its ultimate position may be said to touch the curve at  $Q$  and cut it in another point indefinitely near to the point  $Q$ .

We have therefore the following definitions:—

**Def.** If a circle touch a curve at a point  $P$  and cut the curve at an adjacent point  $Q$ , and if the radius of the circle be made to change in such a way that the point  $Q$

moves up to and ultimately coincides with  $P$ , then the circle in its ultimate position is called the **circle of curvature** of the curve at the point  $P$ .

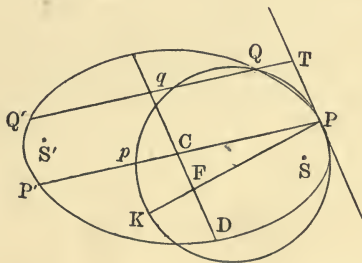
Also the centre of the circle is called the **centre of curvature** of the curve at  $P$ , and the chord of the circle in any direction is called the **chord of curvature** of the curve in that direction.

**145.** *To find the chord of curvature at any point of a central conic in any direction.*

Let a circle be drawn touching the conic at the given point  $P$  and cutting it at an adjacent point  $Q$ .

Let the diameter  $PCP'$  cut the circle in  $p$ , and let a line through  $Q$  parallel to  $PCP'$  cut the conic again in  $Q'$ , the circle in  $q$ , and the tangent at  $P$  in  $T$ .

Then when  $Q$  moves up to and ultimately coincides with  $P$ ,  $Pp$  will ultimately be the chord of curvature at  $P$  through the centre of the conic.



Now from the conic we have

$$CP^2 : CD^2 = TQ \cdot TQ' : TP^2,$$

where  $CD$  is the semi-diameter conjugate to  $CP$ .

But, since  $TP$  touches the circle,  $TP^2 = TQ \cdot Tq$ .

$$\begin{aligned} \text{Hence } CP^2 : CD^2 &= TQ \cdot TQ' : TQ \cdot Tq \\ &= TQ' : Tq. \end{aligned}$$

But when the circle becomes the circle of curvature at  $P$ , the chords  $QQ'q$  and  $PP'p$  will coincide, so that  $TQ' = PP'$  and  $Tq = Pp$  ultimately.

Hence we have ultimately

$$CP^2 : CD^2 = PP' : Pp = 2CP : Pp ;$$

$$\therefore Pp \cdot CP = 2CD^2 \dots\dots\dots (i).$$

Thus *the chord of curvature through the centre of the conic is equal to  $2CD^2/CP$ .*

If  $PK$  be the diameter of the circle, it will cut  $CD$  at right angles in the point  $F$ , and  $KpP$  will also be a right angle.

$$\begin{aligned} \text{Hence} \quad PK \cdot PF &= PC \cdot Pp \\ &= 2CD^2, \text{ from (i).} \end{aligned}$$

Thus *the diameter of curvature is equal to  $2CD^2/PF$ .*

Again, if  $Pl$  be the chord of curvature in any other direction, and if  $Pl$  cut  $CD$  in the point  $L$ ; then, since  $PlK$  is a right angle

$$Pl \cdot PL = PF \cdot PK = 2CD^2.$$

If  $Pl$  pass through a focus of the conic, we know that  $PL = CA$ , so that *the chord of curvature through a focus of the conic is equal to  $2CD^2/CA$ .*

**146.** *To find the chord of curvature at any point of a parabola in any direction.*

Let a circle be drawn touching the parabola at the given point  $P$  and cutting it at an adjacent point  $Q$ .

Let the diameter through  $P$  cut the circle in  $p$ , and let the diameter through  $Q$  cut the circle in  $q$  and the tangent at  $P$  in  $T$ .

Let  $QV$  be the ordinate to the diameter  $PVp$  of the parabola.

Then, since  $TP$  touches the circle,

$$TP^2 = TQ \cdot Tq.$$



chord  $QR$  make equal angles with an axis. Also, if the radius of the circle be changed so that the point  $Q$  moves up to and ultimately coincides with  $P$ , then the tangent at  $P$  and the chord  $PR$  will make equal angles with an axis.

Thus *the tangent at any point  $P$  of a conic and the common chord of the conic and its circle of curvature at  $P$  make equal angles with an axis.*

The common chord of a conic and its circle of curvature at any given point can therefore be at once drawn.

### EXAMPLES.

1. Find a point  $P$  on a parabola such that the circle of curvature at  $P$  will pass through the other extremity of the focal chord through  $P$ .

2. Prove that the distance of the centre of curvature at any point of a parabola from the directrix is three times that of the point.

3. Shew that the common chord of a parabola and the circle of curvature at any point on the curve is a tangent to another fixed parabola.

4. Find the points on a parabola the circles of curvature at which pass through a fixed point on the curve.

5. Having given a point on a parabola and the circle of curvature of the curve at that point, shew that the focus must lie on a fixed circle and that the directrix must pass through a fixed point.

6. Find a point  $P$  on an ellipse such that the circle of curvature at  $P$  will pass through the other extremity of the diameter through  $P$ .

7. Shew that the diameter of curvature of a conic at an extremity of its transverse axis is equal to the latus-rectum.

8.  $P$  is any point on a parabola whose vertex is  $A$ , and  $PM$ ,  $PN$  are the perpendiculars on the directrix and axis respectively. Shew that, if  $MA$  and  $PN$  intersect in the point  $O$ , the diameter through  $O$  will pass through the centre of curvature at  $P$ .

9. Shew that, if  $S$  be the focus of a parabola,  $P$  any point on the curve and  $O$  the centre of curvature at that point; then

$$PO^2 = SO^2 + 3PS^2.$$

10. Shew that, at any point  $P$  of a conic the chord of curvature through a focus is equal to the focal chord of the conic which is parallel to the tangent at  $P$ .

11. The normal at  $P$  to a rectangular hyperbola cuts the curve again in  $Q$ ; shew that  $PQ$  is equal to the diameter of curvature at  $P$ .

12. Prove that the circle of curvature of an ellipse at  $P$ , one of the extremities of an equi-conjugate diameter, cuts the auxiliary circle so that their common chord is parallel to the diameter through  $P$ .

13. Construct an ellipse having given one focus, one point on the curve and the circle of curvature at that point.

14. Shew that the centre of curvature at any point  $P$  of an ellipse is the pole of the tangent at  $P$  with respect to the confocal hyperbola through  $P$ .

15. Shew that, if the circle of curvature at any point  $P$  on an ellipse cut the curve again in  $O$ , there will be two other points  $Q, R$  on the ellipse the circles of curvature at which will pass through  $O$ , and the four points  $P, Q, R, O$  will lie on a circle.

16. The circle of curvature of a parabola at  $P$  meets the curve again in  $Q$ , and the tangent at  $P$  cuts the axis in  $T$ ; prove that

$$PQ = 4PT.$$

17. Prove that, if the circle of curvature at any point on an ellipse pass through a focus, the point must lie midway between the minor axis and a directrix.

18. Three points  $A, P, B$  are taken on an ellipse whose centre is  $C$ . Parallels to the tangents at  $A, B$  drawn from  $P$  meet  $CB$  and  $CA$  respectively in the points  $Q$  and  $R$ . Shew that  $QR$  is parallel to the tangent at  $P$ .

19. Two tangents  $TP, TQ$  are drawn to an ellipse and any chord  $TRS$  is drawn,  $V$  being the middle point of the intercepted part;  $QV$  meets the ellipse in  $P'$ ; prove  $PP'$  is parallel to  $ST$ .

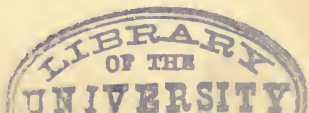
20. Shew that the line joining the points of contact of any two parallel tangents, one to each of two given similar and similarly situated ellipses, will pass through one or other of two fixed points.

21. Shew that, if  $PP', DD'$  and  $pp', dd'$  are two pairs of conjugate diameters of an ellipse, the lines  $Pp, Pp'$  are parallel respectively to  $D'd, D'd'$ .

22. Through the two fixed points  $A, B$  of an ellipse the chords  $AP, BQ$  are drawn parallel to each other. Shew that  $PQ$  also touches a similar and similarly situated ellipse.

23.  $A$  and  $B$  are any two points such that the polar of  $A$  with respect to a given ellipse passes through  $B$ . From  $D$ , the middle point of  $AB$ , a tangent  $DP$  is drawn to the ellipse. Shew that, if  $CQ, CR$  are the semi-diameters parallel to  $AB$  and  $DP$ , then

$$AB : CQ = 2DP : CR.$$



## CHAPTER VII.

### CROSS-RATIOS AND INVOLUTION.

#### ANHARMONIC PROPERTIES OF CONICS.

**148.** A SET of points on a straight line is called a **range**; and a set of straight lines passing through a point is called a **pencil**, each line being called a **ray** of the pencil.

If  $P, Q, R, S$  be four points on a straight line, the ratio  $\frac{PQ}{RQ} : \frac{PS}{RS}$  or  $PQ \cdot RS : PS \cdot RQ$ , regard being paid to the *directions* of the segments [Art. 102], is called the **anharmenic ratio** or the **cross-ratio** of the range  $P, Q, R, S$ , and is expressed by the notation  $\{PQRS\}$ .

**149.** When a straight line  $PR$  is divided internally in  $Q$  and externally in  $S$  in the same ratio, it is said to be divided **harmonically**; and  $Q$  and  $S$  are said to be *harmonically conjugate* with respect to  $P$  and  $R$ .

Thus  $P, Q, R, S$  is a harmonic range if

$$PQ : QR = PS : RS,$$

or

$$PQ : -RQ = PS : RS.$$

Hence the cross of a harmonic range is equal to  $-1$ .

When

$$\{PQRS\} = -1, \quad PQ \cdot RS = -RQ \cdot PS;$$

$$\therefore PQ : PS = PR - PQ : PS - PR,$$

so that  $PQ, PR, PS$  are in harmonical progression.

If  $\{PQRS\} = -1$ , or  $PQ \cdot RS = PS \cdot QR$  and  $V$  be the middle point of  $PR$ ; then we have

$$(PV + VQ)(VS - PV) = (PV + VS)(PV - VQ),$$

whence

$$VQ \cdot VS = VP^2 = VR^2.$$

Similarly if  $W$  be the middle point of  $QS$

$$WR \cdot WP = WS^2 = WQ^2.$$

**150.** The definition of the cross-ratio of four points on a straight line requires that the points should be taken in a particular order. It follows, however, at once from the definition that

$$\{PQRS\} = \{QPSR\} = \{RSPQ\} = \{SRQP\}.$$

Hence *the cross-ratio of four points is unaltered if any two of the points be interchanged and the other two be also interchanged.*

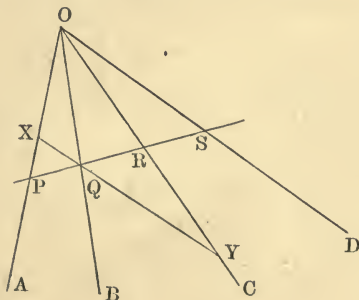
By means of the relation

$$PQ \cdot RS + PR \cdot SQ + PS \cdot QR = 0,$$

which is true for *all positions* of  $P, Q, R, S$  on a straight line, it can be shewn that if  $\{PQRS\} = x$ , the different values of the cross-ratios, obtained by taking the four points in every possible order, can be shewn to be

$$x, \frac{1}{x}, 1-x, \frac{1}{1-x}, \frac{x}{1-x} \text{ and } \frac{1-x}{x}.$$

**151.** *If a pencil of four straight lines  $OA, OB, OC, OD$  be cut by any straight line in the points  $P, Q, R, S$  respectively; then will  $\{PQRS\}$  be constant.*



Through  $Q$  draw  $XQY$  parallel to  $OD$  and cutting  $OA, OC$  in  $X, Y$  respectively.

Then  $PQ : PS = XQ : OS,$

and  $RS : RQ = OS : YQ;$

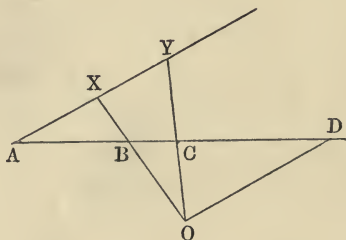
$$\therefore PQ \cdot RS : PS \cdot RQ = XQ : YQ.$$

But  $XQ : YQ$  is obviously constant for all positions of  $Q$ , and hence  $\{PQRS\}$  is constant for all positions and directions of the transversal.

**Def.** The cross-ratio of a pencil of four lines  $OA, OB, OC, OD$  is the cross-ratio of the range in which the lines are cut by any transversal, and is expressed by the notation  $O\{ABCD\}$ .

**Ex. 1.** Having given three points on a straight line, find a fourth point on that line such that the range may have a given cross-ratio.

Let  $A, B, C$  be the three given points; draw any line  $AXY$  through  $A$ , and take the points  $X, Y$  upon it such that  $AX : YX$  is equal to the given cross-ratio.



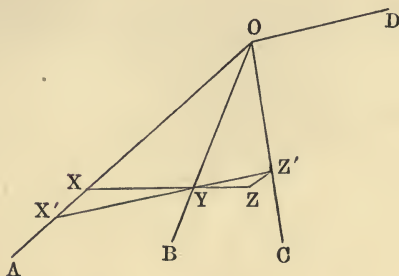
Let  $XB, YC$  meet in  $O$ , and draw through  $O$  a line parallel to  $AXY$  cutting  $ABC$  in  $D$ . Then  $D$  will be the point required.

For  $\{ABCD\} = \{AXY\infty\} = AX : YX.$

**Ex. 2.** Having given three lines meeting in a point, find a fourth line through that point such that the pencil may have a given cross-ratio.

Let  $OA, OB, OC$  be the given straight lines. Draw any line cutting  $OA$  in  $X$  and  $OB$  in  $Y$ , and take the point  $Z$  on this line such

that  $XY : ZY$  is equal to the given cross-ratio. Through  $Z$  draw a line parallel to  $OA$  cutting  $OC$  in  $Z'$ , and let  $Z'Y$  cut  $OA$  in  $X'$ .



Then the line  $OD$  parallel to  $X'Z'$  will be the line required.

For  $O\{ABCD\} = \{X'YZ'\infty\} = X'Y : Z'Y = XY : ZY$ .

**Ex. 3.** *If two ranges of equal cross-ratio, on different straight lines, have one common point, the lines joining their other common points will be concurrent.*

For, if  $\{ABCD\} = \{AB'C'D'\}$ , and if  $BB'$ ,  $CC'$  intersect in  $O$ , and  $OD$  cut  $AB'C'D'$  in  $X$ . Then  $\{AB'C'X\} = \{ABCD\} = \{AB'C'D'\}$ , whence it follows that  $X$  coincides with  $D'$ .

**Ex. 4.** *If two pencils of equal cross-ratio through different points have one common ray, the points of intersection of their other corresponding rays are collinear.*

If  $O\{ABCD\} = O'\{AB'C'D'\}$ , and if  $X, Y, Z$  be the three points of intersection of the other corresponding rays. Then, if  $XY$  does not pass through  $Z$ , let it cut  $OO'$  in  $P$  and  $OD, O'D'$  respectively in  $K, K'$ . Then

$O\{ABCD\} = \{PXYK\}$ , and  $O'\{AB'C'D'\} = \{PXYK'\}$ .

Thus  $\{PXYK\} = \{PXYK'\}$ , which is impossible unless  $K$  and  $K'$  coincide with  $Z$ .

**Ex. 5.** *Each of the three diagonals of a quadrilateral is divided harmonically by the other two diagonals.*

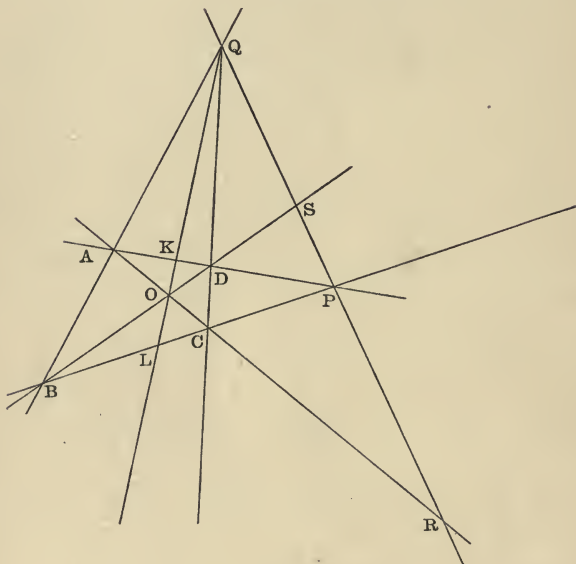
Let the straight lines  $QAB, QDC, PDA$  and  $PCB$  be the sides of the quadrilateral. The line joining the point of intersection of two of these lines with the point of intersection of the other two is called a *diagonal* of the quadrilateral. There are therefore three diagonals, viz.  $PQ, AC, BD$ , in the figure.

We have to prove that

$$\{AOCR\} = \{BODS\} = \{QSPR\} = -1.$$

Let  $QO$  cut  $AD$  in  $K$  and  $BC$  in  $L$ .

$$\begin{aligned} \text{Then } \{AOCR\} &= Q \{AOCR\} = \{AKDP\} \\ &= O \{AKDP\} = \{CLBP\} \\ &= Q \{CLBP\} = \{COAR\}. \end{aligned}$$



And, since  $\{AOCR\} = \{COAR\}$ ,

$$\frac{AO \cdot CR}{AR \cdot CO} = \frac{CO \cdot AR}{CR \cdot AO}; \quad \therefore \{AOCR\} = \pm 1.$$

We must take the *negative* sign, for it is easily seen that two of the rays coincide if the anharmonic ratio of a pencil be equal to  $+1$ .

Hence the diagonal  $AC$  is cut harmonically.

We can prove in a similar manner that the other diagonals are divided harmonically.

## INVOLUTION.

**152. Def.** When several pairs of points  $A, A'$ ;  $B, B'$ ;  $C, C'$ ; &c. lying on a straight line are such that their distances from a fixed point  $O$  on the line are connected by the relations  $OA \cdot OA' = OB \cdot OB' = OC \cdot OC' = \dots$ , the points are said to form a range in **involution**, of which the fixed point  $O$  is called the **centre**.

Two corresponding points, such as  $A, A'$ , are said to be *conjugate* to one another. The point conjugate to the centre is at an infinite distance.

If each point be on the same side of the centre as its conjugate, there will be two points  $K_1, K_2$ ; one on each side of the centre, such that  $OK_1^2 = OK_2^2 = OA \cdot OA'$ . These points  $K_1, K_2$  are called **double points** or **foci**.

When the points of a conjugate couple are on opposite sides of the centre, the double points are imaginary.

**153.** *An involution is completely determined when two pairs of conjugate points are given.*

For, if any two circles be drawn through the points  $A, A'$  and  $B, B'$  respectively, the radical axis of the circles will meet the line  $AA'BB'$  in a point  $O$  such that

$$OA \cdot OA' = OB \cdot OB',$$

and there is only one such point.

**154.** *If any number of points be in involution the cross-ratio of any four points is equal to that of their four conjugates.*

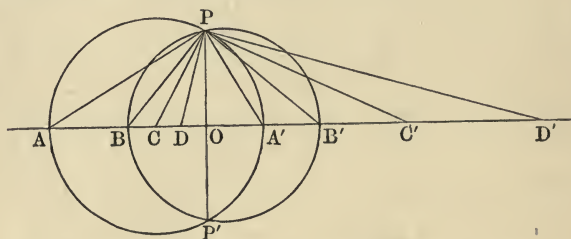
Let the pairs of conjugate points be  $A, A'$ ;  $B, B'$ ;  $C, C'$  and  $D, D'$ .

Let the radical axis of the circles on  $AA'$  and  $BB'$  as diameters cut the line  $ABCD$  in the point  $O$ ; then  $O$  is the centre of the involution.

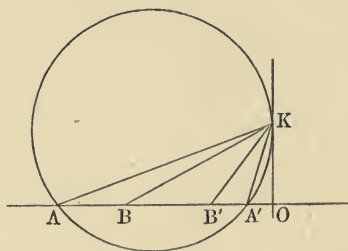
If these circles intersect in real points  $P, P'$ ; then  $APA'$ ,  $BPB'$  and also  $AOP$  are right angles.

Hence  $AO \cdot OA' = OP^2 = BO \cdot OB' = CO \cdot OC' = \&c.$ , the angles  $COC'$ ,  $DOD'$  are therefore also right angles.

Hence the angles  $APB$ ,  $A'PB'$  are equal; also the angles  $BPC$ ,  $B'PC'$  and the angles  $CPD$ ,  $C'PD'$  are equal. The pencils formed by joining  $P$  to the four points  $A, B, C, D$  is therefore equiangular to the pencil formed by joining  $P$  to the four points  $A', B', C', D'$ , whence it follows that  $P\{ABCD\} = P\{A'B'C'D'\}$ .



If, however, the circles whose diameters are  $AA'$  and  $BB'$  do not meet in real points, which will be the case when two conjugate points are on the same side of the centre  $O$ , draw a circle through  $A, A'$  so as to touch the line through  $O$  perpendicular to  $AA'$ ; and let  $K$  be the point of contact.



Then since  $OK^2 = OA \cdot OA' = OB \cdot OB'$ , it follows that  $OK$  touches in  $K$  a circle through  $B$  and  $B'$ , and similarly for the other pairs of points.

Hence the angles  $OKA'$ ,  $OAK$  are equal, and also the angles  $OKB'$ ,  $OBK$  are equal; therefore the angles  $A'KB'$ ,  $AKB$  are equal.

Thus the pencil formed by joining  $K$  to  $A, B, C, D$  is equiangular to the pencil formed by joining  $K$  to  $A', B', C', D'$ . The two pencils are therefore equi-cross.

It should be noticed that it has been incidentally proved that *if two pairs of conjugate points of a range in involution subtend a right angle at any point, every pair will subtend a right angle at that point.*

**Cor. I.** From the above we obtain a necessary and sufficient condition that three pairs of points may be in involution, namely the condition

$$\{ABCA'\} = \{A'B'C'A\}.$$

**Cor. II.** Any pair of conjugate points in an involution form a harmonic range with the two double points.

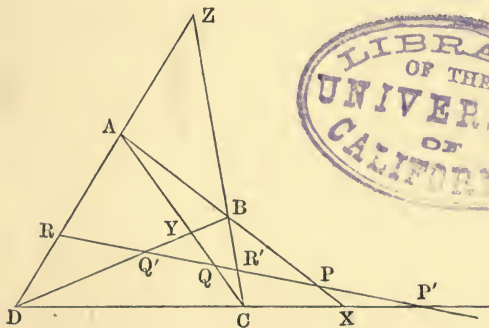
For, if the double points be  $K_1, K_2$ , and  $A, A'$  be a pair of conjugate points, we have

$$\{K_1AK_2A'\} = \{K_1A'K_2A\}.$$

**Ex. 1.** Any transversal cuts the three pairs of opposite sides of any quadrangle in three pairs of points in involution.

Let  $A, B, C, D$  be the angular points of the quadrangle. Let  $AB$  and  $CD$  meet in  $X$ ,  $AC$  and  $BD$  in  $Y$ , and  $AD$  and  $BC$  in  $Z$ .

Let any line cut these pairs of opposite sides in  $P, P'$ ;  $Q, Q'$  and  $R, R'$ .



$$\begin{aligned}
 \text{Then} \quad \{PQRP'\} &= A \{PQRP'\} \\
 &= \{XCDP'\} \\
 &= B \{XCDP'\} \\
 &= \{PR'Q'P'\} = \{P'Q'R'P'\}.
 \end{aligned}$$

Ex. 2. *The three pairs of lines from any point to the extremities of the three diagonals of any quadrilateral are in involution.*

**155. Def.** If any number of pairs of points in involution be joined to any point  $O$ , the pencil so obtained is said to be in involution.

If  $A, A'; B, B'; C, C';$  &c. be pairs of points in involution, and if the pencil formed by joining these points to  $O$  be cut by any other transversal in the pairs of points  $a, a'; b, b'; c, c';$  &c.

Then since  $A, A'; B, B';$  &c. are pairs of points in involution we have relations of the form

$$\{ABCA'\} = \{A'B'C'A\}.$$

$$\text{But} \quad \{ABCA'\} = \{abca'\},$$

$$\text{and} \quad \{A'B'C'A\} = \{a'b'c'a\}.$$

Hence we have

$$\{abca'\} = \{a'b'c'a\}, \text{ \&c.,}$$

whence it follows that  $a, a'; b, b'; c, c';$  &c. are pairs of points in involution.

Thus, *if a pencil be cut by any transversal in pairs of points in involution, it will be cut by any other transversal in pairs of points which are in involution.*

**Cor. I.** *Pairs of perpendicular lines through a point are cut in involution by a straight line.*

**Cor. II.** *Pairs of conjugate diameters of a conic are in involution.*

We know that pairs of conjugate diameters of a conic are cut by any tangent in pairs of points which are in involution, the point of contact of that tangent being the centre of the involution [Art. 108].

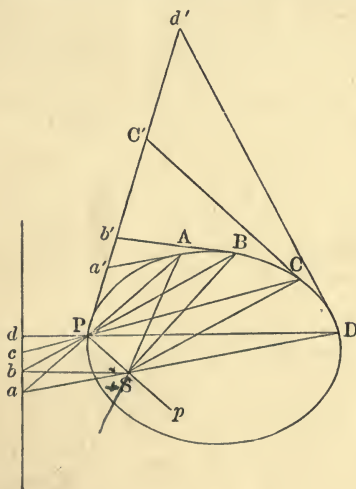
Conjugate diameters are therefore cut by *any* straight line in pairs of points in involution, the asymptotes of the conic being the double lines of the involution.

### ANHARMONIC PROPERTIES OF CONICS.

**156.** *The cross-ratio of the pencil formed by joining any point on a conic to four fixed points is constant and equal to that of the range in which the tangents at those points are cut by any other tangent.*

Let  $A, B, C, D$  be four fixed points on a conic whose focus is  $S$ , and let  $P$  be any other point on the curve. Let  $PA, PB, PC, PD$  cut the directrix corresponding to the focus  $S$  in  $a, b, c, d$  respectively; and let  $PS$  be produced to  $p$ . Then we know that  $Sa$  will bisect the angle  $PSA$  or  $ASp$  according as  $P$  and  $A$  are on opposite branches or on the same branch of the curve.

From this it follows that, for all positions of  $P$ , the angle  $aSb$  is constant, being equal or complementary to



half the angle  $ASB$  according as  $A$  and  $B$  are on the same or on opposite branches of the curve. Since the angles  $aSb$ ,  $bSc$ ,  $cSd$  are constant angles, it follows that  $S\{abcd\}$  is constant.

$$\begin{aligned}\text{But} \quad S\{abcd\} &= \{abcd\} \\ &= P\{abcd\} \\ &= P\{ABCD\}.\end{aligned}$$

Thus  $A, B, C, D$  subtend a pencil of constant cross-ratio at any point on the conic.

Next, let the tangent at  $P$  cut the tangents at  $A, B, C, D$  in  $a', b', c', d'$  respectively.

Then we know [Art. 17] that  $Sa', Sb', Sc', Sd'$  are perpendicular respectively to  $Sa, Sb, Sc, Sd$ , and therefore

$$S\{a'b'c'd'\} = S\{abcd\}.$$

$$\begin{aligned}\text{Hence} \quad \{a'b'c'd'\} &= S\{a'b'c'd'\} \\ &= S\{abcd\} = P\{abcd\} \\ &= P\{ABCD\}.*\end{aligned}$$

**157.** The preceding proposition enables us to construct a conic through five given points, or touching five given straight lines.

If  $A'$  be a point on the curve indefinitely near to  $A$ ; we have  $A'\{ABCD\} = E\{ABCD\}$ .

Thus the cross-ratio of the pencil  $A'\{ABCD\}$  is known; and  $A'B, A'C, A'D$  ultimately coincide with  $AB, AC, AD$  respectively. Hence the tangent at  $A$  is the line  $AX$  through  $A$  which is such that  $A\{XBCD\} = E\{ABCD\}$ . We can therefore draw the tangents at the points  $A, B, C, D$ .

Then, if the tangents at  $A$  and  $B$  meet in  $X$ , the line through  $X$  and  $V$ , the middle point of  $AB$ , will pass through the centre of the curve.

Also, if the tangents at  $B$  and  $C$  intersect in  $Y$ , the line through  $Y$  and  $W$ , the middle point of  $BC$ , will pass

\* This proof is due to Mr B. W. Horne, late Fellow of St John's College.

through the centre. Thus the centre of the conic is determined.

If  $O$  be the centre of the conic, and a line through  $O$  parallel to  $AB$  cut the tangent at  $B$  in  $T$ ; then, since  $OX$ ,  $OT$  are conjugate diameters, if  $X$ ,  $T$  be on opposite sides of  $B$ , the conic must be an ellipse and the square of the diameter conjugate to  $OB$  will be equal to  $XB \cdot BT$  [Art. 72]; if, however,  $X$  and  $T$  are on the same side of  $B$ , the conic must be a hyperbola, and the asymptotes will cut the tangent at  $B$  in the points  $L$ ,  $L'$  such that

$$LB^2 = BL'^2 = BX \cdot BT \text{ [Art. 108].}$$

In the first case we have a pair of conjugate diameters of an ellipse given in position and magnitude, and the axes &c. can be found as in Art. 77. In the second case we have the asymptotes and a tangent, and the axes &c. can be found as in Art. 114.

Again, let  $AB$ ,  $BC$ ,  $CD$ ,  $DE$  and  $EA$  be five given tangents to a conic, and let  $AB$ ,  $EA$ ,  $CD$ ,  $BC$  cut  $DE$  in the points  $L$ ,  $E$ ,  $D$ ,  $N$  respectively, also let  $AB$  cut  $DC$  in  $K$ .

Then, if  $A'B'$  be a tangent nearly coincident with  $AB$ , and if this tangent cuts the tangents  $AB$ ,  $EA$ ,  $CD$ ,  $BC$  in the points  $F$ ,  $A'$ ,  $K'$ ,  $B'$  respectively, we shall have

$$\{FA'K'B'\} = \{LEDN\}.$$

Now let  $A'B'$  move up to and ultimately coincide with  $AB$ , then  $K'$  will ultimately coincide with  $K$ ,  $B'$  with  $B$ ,  $A'$  with  $A$ , and  $F$  with the point of contact of the tangent  $AB$ .

Thus 
$$\{AFBK\} = \{LEDN\},$$

and as the cross-ratio of the range  $A$ ,  $F$ ,  $B$ ,  $K$  is known, and three of the points are known, the fourth point which is the point of contact of  $AB$  can be at once found. Having found the points of contact of the five tangents, the construction can be completed as in the former case.

From the above constructions it is clear that *one* conic and only one will pass through five given points, no four of which are on a straight line, and *one* conic and only one will touch five given straight lines no four of which pass through a point.

**158.** *The locus of a point which moves so that the pencil formed by joining it to four fixed points not on a straight line is of constant cross-ratio is a conic through the four given points.*

Let  $A, B, C, D$  be the four given points, and let  $P, Q$  be any two points such that

$$P \{ABCD\} = Q \{ABCD\}.$$

By the preceding Article one and only one conic will pass through the five points provided no four of the points are on a straight line.

Hence, if  $Q$  be not on the conic determined by the five points  $A, B, C, D, P$ , let  $QA$  cut this conic in  $R$ .

Then, since  $R$  is on the conic through  $A, B, C, D$  and  $P$ ,

$$\begin{aligned} R \{ABCD\} &= P \{ABCD\} \\ &= Q \{ABCD\}, \end{aligned}$$

whence it follows that  $B, C$  and  $D$  must be on a straight line; and similarly it can be shewn that  $A, B$  and  $C$  are on a straight line.

Hence  $Q$  and  $R$  must coincide, for by supposition  $A, B, C$  and  $D$  are not on a straight line.

This proves the proposition.

**159.** *The envelope of a straight line which cuts four fixed non-concurrent straight lines in a range of constant cross-ratio is a conic which touches the four fixed straight lines.*

Let the four fixed straight lines be cut by two other lines in the points  $P, Q, R, S$  and  $P', Q', R', S'$  respectively.

Then one conic and only one will touch the four fixed straight lines and the line  $PQRS$ ; and, if  $P'Q'R'S'$  do not touch this conic, draw another tangent to the conic from  $S'$  and let it cut  $PP'$ ,  $QQ'$ ,  $RR'$  in  $K$ ,  $L$ ,  $M$  respectively.

Then, by Art. 156,

$$\begin{aligned}\{KLMS'\} &= \{PQRS\} \\ &= \{P'Q'R'S'\},\end{aligned}$$

whence it follows that  $P'KP$ ,  $Q'LQ$ ,  $R'MR$  meet in a point; and it can be similarly shewn that  $P'P$ ,  $Q'Q$ ,  $S'S$  will meet in a point.

Since the four given straight lines do not meet in a point, it follows that the lines  $P'Q'R'S'$  and  $KLMS'$  must coincide.

**160.** *If any chord of a conic be drawn through a fixed point  $O$ , it will be cut harmonically by the curve and the polar of  $O$ .*

If  $O$  be without the conic, the polar of  $O$  will cut the conic; let the points of intersection be  $A$ ,  $B$ . Let any chord through  $O$  cut the conic in  $Q$ ,  $R$  and the polar of  $O$  in  $V$ , and let the tangents at  $Q$ ,  $R$  meet in  $T$ .

Then, if  $A'$ ,  $B'$  be points on the conic very near to  $A$ ,  $B$  respectively, we have

$$A' \{AQBR\} = B' \{AQBR\}.$$

Now when  $A'$ ,  $B'$  move up to and ultimately coincide with  $A$ ,  $B$ , the above pencils will ultimately be cut by the line  $OQVQ'$  in the ranges  $\{OQVR\}$  and  $\{VQOR\}$  respectively.

Hence 
$$\{OQVR\} = \{VQOR\},$$

whence it follows that  $QR$  is divided harmonically in  $O$  and  $V$ .

Since  $OT$  is the polar of the *internal* point  $V$ , the proposition is true for any point, whether external or internal.

**Conversely**, if through any point  $O$  a line be drawn cutting a conic in the points  $Q$ ,  $R$ , and  $V$  be taken on this line such that  $\{OQVR\} = -1$ ; then will  $V$  lie on the polar of  $O$  with respect to the conic.

**161.** *The cross-ratio of a range of four points on a straight line is equal to that of the pencil formed by their polars with respect to any conic.*

Let  $A, B, C, D$  be any four points on a straight line; then the polars of these points with respect to any conic will all pass through the pole of  $ABCD$  with respect to that conic [Art. 115].

Let  $PA', PB', PC', PD'$  be the polars of  $A, B, C, D$  with respect to a conic whose focus is  $S$ , and let the polars cut the corresponding directrix in the points  $a, b, c, d$  respectively.

Then, since the angles  $aSA, bSB, cSC, dSD$  are all right angles [Art. 17], it follows that

$$S\{ABCD\} = S\{abcd\}.$$

$$\text{But } S\{abcd\} = \{abcd\} = P\{abcd\} = P\{A'B'C'D'\}.$$

$$\text{Hence } \{ABCD\} = P\{A'B'C'D'\}.$$

**162. Def.** Two points are said to be **conjugate points** with respect to a conic when each lies on the polar of the other; also two straight lines are said to be **conjugate lines** with respect to a conic when each passes through the pole of the other.

Conjugate diameters of a conic are conjugate lines through the centre of the conic.

*Pairs of conjugate lines with respect to a conic which pass through a point are in involution, the tangents from the point to the conic being the double lines of the involution.*

Let  $PA, PA'; PB, PB';$  &c. be any number of pairs of conjugate lines with respect to any conic.

Let the polar of  $P$  with respect to the conic cut  $PA', PB', PC', PD'$  in the points  $A', B', C', D'$  respectively; then  $A'$  is the pole of  $PA$ , for the pole of  $PA$  is by supposition on  $PA'$  and it must also be on the polar of  $P$ . Thus  $A', B', C', D'$  are the poles of  $PA, PB, PC, PD$  respectively.

Hence, by the preceding Article,

$$P\{ABCD\} = \{A'B'C'D'\} = P\{A'B'C'D'\},$$

whence it follows [Art. 154] that  $PA, PA'; PB, PB'; PC, PC';$  &c. are pairs of lines in involution, and it is obvious that the tangents from  $P$  are the double lines of the involution.

It can also be proved in a similar manner that *pairs of conjugate points on a straight line are in involution, the points in which the straight line cuts the conic being the double points of the involution.*

Ex. 1. *Through any point  $P$  on a conic the chords  $PQ, PQ'$  are drawn making equal angles with the tangent at  $P$ . Shew that  $QQ'$  passes through a fixed point.*

Let  $QQ'$  cut the tangent at  $P$  in the point  $T$ , and let the polar of  $T$  cut  $QQ'$  in  $V$ . Then the range  $T, Q, V, Q'$  is harmonic, and  $TP$  bisects the exterior angle between  $PQ$  and  $PQ'$ , whence it follows that  $PV$  bisects the angle  $QPQ'$ , so that  $PV$  is the normal at  $P$ . Hence  $T$  is a fixed point, namely the pole of the normal chord through  $P$ .

Ex. 2. *All conics through four given points have a common self-polar triangle.*

Let  $A, B, C, D$  be the four given points [see figure, p. 208].

Then, since  $\{BODS\} = -1$ , it follows that  $S$  is on the polar of  $O$  with respect to any conic of the system.

Similarly  $R$  is on the polar of  $O$  with respect to any conic of the system.

Hence  $O$  is the pole of the line  $SR$ .

Again, since  $\{AKDP\} = -1$ , it follows that  $K$  is on the polar of  $P$  with respect to any conic of the system.

Similarly  $L$  is on the polar of  $P$ .

Hence  $P$  is the pole of the line  $LOKQ$  with respect to any conic of the system.

And, since the polars of  $O$  and  $P$  both pass through  $Q$ ,  $Q$  must be the pole of the line  $OP$ .

Thus the triangle  $OPQ$  is such that each angular point is the pole of the opposite side with respect to any conic through the four points  $A, B, C, D$ .

Ex. 3. *All conics touching four given straight lines have a common self-polar triangle.*

Let  $AB, BC, CD, DA$  be the four given straight lines [figure, p. 208].

Let  $X$  be the pole of  $AC$  with respect to any conic touching the four given straight lines. Then by Art. 160 the pencil

$$A \{BCDX\} = -1 = C \{BADX\},$$

whence it follows that  $X$  must lie on  $BD$ , and must coincide with the point  $S$ , for we know that  $\{BODS\} = -1$ .

Thus  $S$  is the pole of  $AC$  with respect to any conic of the system.

Similarly  $R$  is the pole of  $BD$ , and  $O$  is the pole of  $PQ$ .

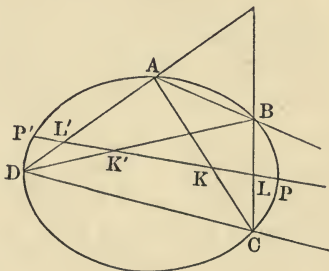
Hence the triangle  $OSR$  is such that each angular point is the pole of the opposite with respect to any conic which touches the four lines  $AB, BC, CD$  and  $DA$ .

Since  $A$  is on the polar of  $S$ , with respect to any conic of the system, the polar of  $A$  will go through  $S$ , that is the line joining the points of contact of the tangents  $AB$  and  $AD$  will pass through  $S$ ; and similarly for any other pair of the tangents.

Hence the line joining any two points of contact of any conic of the system will pass through an angular point of the triangle  $OSR$ .

**163.** *Conics through four given points are cut by any straight line in pairs of points in involution.*

Let  $ABCD$  be the four given points, and let any straight line cut  $AC, BD$  in  $K, K'$  respectively, and  $AD, BC$  in  $L, L'$  respectively. Also let this straight line cut any conic through the four points in  $P, P'$  respectively.



Then  $A \{P'DCP\} = B \{P'DCP\}$ , since the six points lie on a conic.

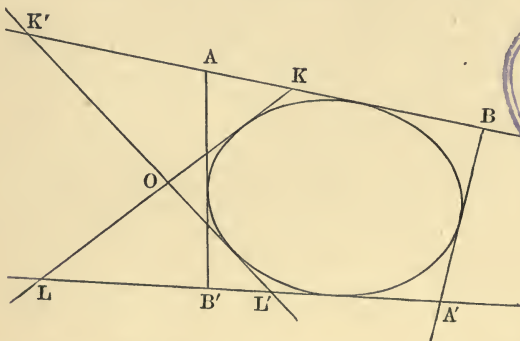
Hence taking the ranges formed by these pencils on the line  $PP'$ , we have

$$\begin{aligned}\{P'L'KP\} &= \{P'K'LP\}; \\ \therefore \{P'L'KP\} &= \{PLK'P'\},\end{aligned}$$

which shews that  $P, P'$  are conjugate points of the involution determined by the two pairs  $K, K'$  and  $L, L'$ .

**164.** *The pairs of tangents drawn from any point to a system of conics touching four given straight lines are in involution.*

Let the four given straight lines be  $AB, BA', A'B', B'A$ . Let the tangents from  $O$  to any conic of the system cut  $AB$  in the points  $K, K'$  and  $A'B'$  in the points  $L, L'$ .



Then, since  $KL, K'L'$  and the four given straight lines touch the same conic,

$$\{BKAK'\} = \{A'LB'L'\}.$$

$$\begin{aligned}\text{Hence } O\{BKAK'\} &= O\{A'LB'L'\} \\ &= O\{A'KB'K'\}; \\ \therefore O\{BKAK'\} &= O\{B'K'A'K\},\end{aligned}$$

which shews that  $OK, OK'$  are conjugate rays of the involution determined by the two pairs  $OA, OA'$  and  $OB, OB'$ .

*Ex. 1. The locus of the centres of conics which pass through four given points is a conic.*

Let  $A, B, C, D$  be the four given points, and let  $U, V, W, X$  be the middle points of  $AB, BC, CD$  and  $DA$  respectively. Let  $O$  be the centre of any conic through the four points, and let  $OP, OQ, OR, OS$  be parallel to  $AB, BC, CD, DA$  respectively. Then  $OU$  and  $OP$  will be conjugate diameters of the conic, and so also will  $OV$  and  $OQ$ ,  $OW$  and  $OR$ , and  $OX$  and  $OS$ .

Hence, as conjugate diameters are conjugate pairs of an involution,  $O\{UVWX\} = O\{PQRS\}$ .

The lines  $OP, OQ, OR, OS$  are fixed in direction, and therefore  $O\{PQRS\}$  is constant.

Hence  $O\{UVWX\}$  is constant, and therefore  $O$  must lie on a fixed conic through  $U, V, W, X$ .

The conic on which the centre lies will also pass through the middle points of  $AC$  and  $BD$ .

Three conics of the system are the line pairs  $AB$  and  $CD$ ,  $AC$  and  $BD$ , and  $AD$  and  $BC$ : the locus of centres must therefore go through the three points of intersection of these pairs of lines.

*Ex. 2. The director-circles of all conics which touch four given straight lines have a common radical axis, and their centres lie on a straight line which passes through the middle points of the diagonals of the quadrilateral formed by the four given straight lines.*

We know [Art. 164] that the tangents drawn to the conics of the system are pairs of lines in involution. If  $O$  be a point of intersection of any two of the director-circles, two pairs of conjugate rays of a pencil in involution will be at right angles; every pair will therefore [Art. 154] be at right angles, so that  $O$  is on the director-circle of every other conic of the system.

Since every director-circle will pass through the common points of any two of them, the director-circles have a common radical axis, and their centres must therefore lie on a straight line perpendicular to this radical axis.

Now the double line joining the extremities of a diagonal of the quadrilateral formed by the given straight lines is a limiting form of a conic which touches the lines. Hence the middle point of a diagonal is on the locus of the centres of the conics of the system. The locus of centres must therefore be the straight line through the middle points of the three diagonals of the quadrilateral.

One of the conics of the system will be a parabola, and the directrix of this parabola will be the common radical axis of the director-circles.

*Ex. 3. Chords of a conic which subtend a right angle at a fixed point on the curve will all intersect on the normal at the point.*

Let  $aa', bb', cc'$  be three chords of a conic which subtend a right angle at the point  $O$  on the curve.

Then  $Oa, Oa'$ ;  $Ob, Ob'$ ;  $Oc, Oc'$  are pairs of perpendicular lines and are therefore in involution.

$$\text{Hence} \quad O \{aba'c\} = O \{a'b'ac'\}.$$

$$\text{But} \quad O \{aba'c\} = c' \{aba'c\},$$

$$\text{and} \quad O \{a'b'ac'\} = b \{a'b'ac'\}.$$

$$\begin{aligned} \text{Hence} \quad c' \{aba'c\} &= b \{a'b'ac'\} \\ &= b \{ac'a'b'\}. \end{aligned}$$

These pencils of equal cross-ratio have one ray coincident, namely  $bc'$ : the intersections of their other corresponding rays must therefore lie on a straight line. Thus  $a, a'$  and the point of intersection of  $cc'$  and  $bb'$  must lie on a line.

Hence  $aa', bb'$  and  $cc'$  meet in a point, and therefore every chord which subtends a right angle at  $O$  must pass through the point of intersection of any two such chords.

The fixed point through which all the chords pass must lie on the normal at  $O$ , for the normal is a limiting position of one of the chords.

**Ex. 4.** *If the perpendicular from a point  $P$  on its polar with respect to a given conic pass through a fixed point  $O$ , the point  $P$  will lie on a rectangular hyperbola whose asymptotes are parallel to the axes of the conic and which passes through the point  $O$  and through the centre of the conic.*

If the line through  $P$  perpendicular to its polar with respect to a conic cut the transverse axis of the conic in  $G$ , and  $PN$  be the perpendicular on that axis, we know that  $CG : CN$  is constant.

Hence, if  $PL, PM$  be parallel to the axes of the conic, the pencil  $P \{CGLM\}$  will be constant.

But since  $P \{COLM\}$  is constant, it follows that  $P$  is on a fixed conic through  $C, O$  and two points at infinity in the direction of the axes. This proves the proposition.

A particular case of the above theorem is the following:—

*The points on a conic the normals at which pass through a fixed point  $O$  lie on a rectangular hyperbola through  $O$  and the centre of the conic, and whose asymptotes are parallel to the axes of the conic.*

**Ex. 5.** *The locus of the pole of a given straight line, with respect to a system of conics through four fixed points, is a conic.*

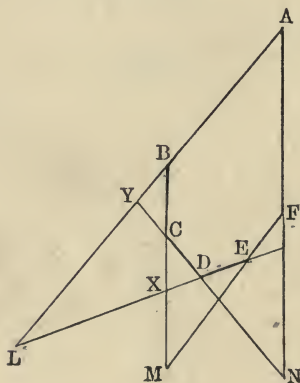
Let the given points be  $A, B, C, D$ , and let the given line cut  $AB, BC, CD, DA$  respectively in  $P, Q, R, S$ . Let  $P'$  be the point on  $AB$  such that  $\{PAP'B\} = -1$ , and let  $Q', R', S'$  be the corresponding points on the other lines. Then the polars of  $P, Q, R, S$  with respect to any conic of the system will go through  $P', Q', R', S'$  respectively. Hence, if  $O$  be the pole of  $PQRS$  with respect to any one of the conics,  $OP, OP'$ , &c. will be pairs of conjugate lines, and therefore  $O \{P'Q'R'S'\} = O \{PQRS\} = \{PQRS\} = \text{constant}$ .

**165. Pascal's Theorem.** *If a hexagon be inscribed in a conic, the three points of intersection of the three pairs of opposite sides lie on a straight line.*

Let  $A, B, C, D, E, F$  be any six points on a conic, and let  $AB$  and  $DE$  intersect in  $L$ ,  $BC$  and  $EF$  in  $M$ , and  $CD$  and  $FA$  in  $N$ . We have to prove that  $L, M, N$  lie on a straight line.

Let  $DE$  cut  $BC$  in  $X$ , and  $DC$  cut  $AB$  in  $Y$ .

$$\begin{aligned}
 \text{Then } L\{BCDM\} &= BCXM \\
 &= E\{BCXM\} \\
 &= E\{BCDF\} \\
 &= A\{BCDF\}, \text{ since the six points} \\
 &\hspace{15em} \text{are on a conic.} \\
 &= \{YCDN\} \\
 &= L\{YCDN\} \\
 &= L\{BCDN\},
 \end{aligned}$$



whence it follows that  $LM$  and  $LN$  are in the same straight line.

Since six points can be taken in order in sixty different ways, there are sixty hexagons corresponding to

six points on a conic ; and, since Pascal's Theorem is true for every one of these hexagons, there are sixty Pascal lines corresponding to six points on a conic.

**166. Brianchon's Theorem.** *If a hexagon be described about a conic, the three diagonals will meet in a point.*

If a hexagon circumscribe a conic, the points of contact of its sides will be the angular points of a hexagon inscribed in the conic. Each angular point of the circumscribed hexagon will be the pole of the corresponding side of the inscribed hexagon ; therefore a diagonal of the circumscribed hexagon, that is a line joining a pair of its opposite angular points, will be the polar of the point of intersection of a pair of opposite sides of the inscribed hexagon. But the three points of intersection of pairs of opposite sides of the inscribed hexagon lie on a straight line by Pascal's Theorem ; hence their three polars, that is the three diagonals of the circumscribed hexagon, will meet in a point.

If we are given *five* tangents to a conic, the points of contact of the tangents can be found by Brianchon's Theorem.

For, let  $A, B, C, D, E$  be the angular points of a pentagon formed by the five given tangents, and let  $K$  be the point of contact of  $AB$  ; then  $A, K, B, C, D, E$  will be the angular points of a circumscribing *hexagon*, two sides of which are coincident. By Brianchon's Theorem,  $DK$  passes through the point of intersection of  $AC$  and  $BE$ , hence  $K$  is found, and the other points of contact can be found in a similar manner.

We can also find the tangents to a conic at five given points on the curve by means of Pascal's Theorem.

For, let  $A, B, C, D, E$  be the five given points, and let  $F$  be the point on the conic indefinitely near to  $A$  ; then, by Pascal's Theorem, the three points of intersection of  $AB$  and  $DE$  ; of  $BC$  and  $EF$  ; and of  $CD$  and  $FA$  lie on a straight line. Hence, if the line joining the point of intersection of  $AB$  and  $DE$  to the point of intersection of  $BC$  and  $EA$  meet  $CD$  in  $H$ , then  $AH$  will be the tangent at  $A$ . The other points of intersection can be found in a similar manner.

**167.** The following examples are important.

**Ex. 1.** *If two triangles circumscribe a conic their six angular points lie on another conic.*

Let  $ABC, A'B'C'$  be the two triangles.

Let  $B'C'$  cut  $AB, AC$  in  $E', D'$  respectively; and let  $BC$  cut  $A'B', A'C'$  in  $E, D$  respectively.

Then the tangents  $BC, B'C'$  will cut the remaining four tangents in ranges of equal cross-ratio.

$$\begin{aligned}\text{Hence} \quad \{BCED\} &= \{E'D'B'C'\}; \\ \therefore A' \{BCED\} &= A \{E'D'B'C'\};\end{aligned}$$

$$\text{i.e.} \quad A' \{BCB'C'\} = A \{BCB'C'\},$$

which proves the proposition.

We can now prove that, *if one triangle can be inscribed in a given conic and circumscribed to another given conic, an infinite number of triangles can be so described.*

For, let  $ABC$  be a triangle inscribed in the conic  $S$  and circumscribed about the conic  $S'$ . Draw any tangent to  $S'$  and let it cut  $S$  in the points  $B', C'$ . Let the other tangents to  $S'$  from  $B'$  and  $C'$  meet in  $A'$ .

Then, we have proved that  $A', B', C', A, B, C$  lie on a conic, and five of these points lie on the conic  $S$ ; the sixth point must therefore lie on  $S$ , for only one conic will go through five given points.

**Ex. 2.** *If two triangles be inscribed in a conic, their six sides touch a second conic.*

Let the triangles be  $ABC$  and  $A'B'C'$ .

Let  $BC$  cut  $A'B', A'C'$  in  $E, F$  respectively, and let  $B'C'$  cut  $AB, AC$  in  $E', F'$  respectively.

Then, since the six points  $A, B, C, A', B', C'$  lie on a conic,

$$\begin{aligned}A \{BCB'C'\} &= A' \{BCB'C'\}; \\ \therefore \{E'F'B'C'\} &= \{BCEF\},\end{aligned}$$

which proves the proposition.

**Ex. 3.** *If two triangles be self-polar with respect to any conic their six angular points lie on a second conic, and their six sides touch a third conic.*

Let  $ABC, A'B'C'$  be two triangles which are self-polar with respect to any conic.

Let  $A'B', A'C'$  cut  $BC$  in  $K, L$  respectively.

Then, since  $B'$  is the pole of  $A'C'$  and  $A$  the pole of  $BC$ ,  $AB'$  will be the polar of  $L$ . Similarly  $AC'$  will be the polar of  $K$ .

Now the pencil formed by any four lines through a point and the range formed by their poles are of equal cross-ratio.

$$\begin{aligned}\text{Hence} \quad A \{BCB'C'\} &= \{CBLK\} \\ &= A' \{CBLK\} \\ &= A' \{CBC'B'\} \\ &= A' \{BCB'C'\},\end{aligned}$$

whence it follows that  $A, B, C, A', B', C'$  lie on a conic.

Again, if  $B'C'$  cut  $AB$ ,  $AC$  in  $F$ ,  $G$  respectively, we have

$$\begin{aligned}\{CBLK\} &= A\{BCB'C'\} \\ &= \{FGB'C'\} \\ &= \{GFC'B'\},\end{aligned}$$

so that  $BC$  and  $B'C'$  cut the other four sides in ranges of equal cross-ratio, whence it follows that the six sides touch the same conic.

We can now prove that, *if one triangle can be inscribed in (or circumscribed about) one given conic and self-polar with respect to another given conic, an infinite number of triangles can be so described.* [See Ex. 1.]

**168. Homographic ranges and pencils.** Ranges and pencils are said to be *homographic* when every four constituents of the one, and the corresponding four constituents of the other, have equal cross-ratios.

**Ex. 1.** *The points of intersection of corresponding lines of two homographic pencils describe a conic.*

Let  $P$ ,  $Q$ ,  $R$ ,  $S$  be any four of the points of intersection of corresponding lines, and  $O$ ,  $O'$  the vertices of the pencils.

Then, by supposition,  $O\{PQRS\} = O'\{PQRS\}$ , whence it follows that  $O$ ,  $O'$ ,  $P$ ,  $Q$ ,  $R$ ,  $S$  lie on a conic. But five points are sufficient to determine a conic; hence the conic through  $O$ ,  $O'$  and any three of the intersections will pass through every other intersection.

**Ex. 2.** *The lines joining corresponding points of two homographic ranges on different straight lines envelope a conic.*

**Ex. 3.**  *$AB$ ,  $A'B'$  are any two finite lines; shew that if  $P$ ,  $P'$  are points on these lines respectively such that  $AP : PB = A'P' : P'B'$ , the line  $PP'$  will envelope a parabola.*

**Ex. 4.** *Find the common lines of two homographic pencils through the same point.*

Let  $OA$ ,  $OB$ ,  $OC$  be any three rays of one pencil and  $OA'$ ,  $OB'$ ,  $OC'$  the three corresponding rays of the other. Then we have to find the line  $OP$  which is such that  $O\{ABCP\} = O\{A'B'C'P\}$ .

Draw any circle through  $O$  and let the points  $A$ ,  $B$ ,  $C$ ,  $A'$ ,  $B'$ ,  $C'$  lie on this circle.

Let  $AB'$ ,  $A'B$  meet in  $X$  and  $AC'$ ,  $A'C$  in  $Y$ ; and let  $XY$  cut the circle at a point  $P$ .

Then, if  $AA'$  cut the line  $XY$  in  $Z$ , we have

$$\{ZXYP\} = A\{ZXYP\} = A\{A'B'C'P\} = O\{A'B'C'P\},$$

since the points are all on the circle; and similarly

$$\{ZXYP\} = O\{ABCP\}.$$

Hence  $OP$  is one of the required straight lines; and from the construction it will be seen that any two homographic pencils will have two real, coincident, or imaginary common rays.

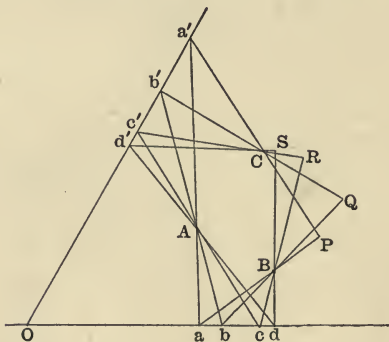
Ex. 5. Find the common points of two homographic ranges on the same straight line.

Join the points to any point  $O$ , and proceed as Ex. in 4.

Ex. 6. The three sides of a triangle pass through fixed points, and the extremities of its base lie on two fixed straight lines; shew that its vertex describes a conic.

Let  $A, B, C$  be the three fixed points, and let  $Oa, Oa'$  be the two fixed straight lines. Suppose triangles drawn as in the figure.

Then the ranges  $\{abcd\ldots\}$  and  $\{a'b'c'd'\ldots\}$  are homographic. Therefore the pencils  $B\{abcd\ldots\}$  and  $C\{a'b'c'd'\ldots\}$  are homographic, and the result follows from Ex. 1.



The above is MacLaurin's method of generating a conic.

Ex. 7. If all the sides of a polygon pass through fixed points, and all the angular points but one move on fixed straight lines; the remaining angular point will describe a conic.

**169. Circular points at infinity.** Since any pair of perpendicular lines through the centre of a circle are conjugate, and since pairs of conjugate lines with respect to a conic which pass through any point are in involution, the real or imaginary tangents from that point to the conic being the double lines of the involution, it follows that the imaginary asymptotes of all circles are parallel, so that *all circles go through the same two imaginary points at infinity*; also concentric circles have common

imaginary asymptotes and therefore have double contact with one another at infinity.

The two imaginary points at infinity through which all circles pass are called **the circular points at infinity**.

Again, since any pair of perpendicular lines through a focus of a conic are conjugate, and since these pairs of conjugate lines are in involution of which the imaginary tangents from the focus are the double lines, it follows that the imaginary tangents to a conic from a focus are parallel to the imaginary asymptotes of any circle, so that *the tangents to any conic through a focus pass through the circular points at infinity*.

Thus all conics with one focus common have two common imaginary tangents through that focus, and confocal conics have four imaginary common tangents.

**170.** We have shewn how to construct the conic which passes through five given points or which touches five given straight lines. The following other cases are of interest.

*Ex. 1. To construct a conic which passes through four given points and touches a given straight line.*

Let the given straight line cut two pairs of opposite sides of the quadrangle formed by the four given points in  $A, A'$  and  $B, B'$ . Then, by Desargue's Theorem, all conics through the four given points are cut in pairs of conjugate points of the involution determined by the pairs  $A, A'$  and  $B, B'$ ; hence if a conic through the four points touch the given line, the point of contact must be one of the double points of the involution. Thus there are *two* (real or imaginary) conics which pass through four given points and touch a given straight line; and since five points on either conic are known the construction can be completed as in Art. 157.

*Ex. 2. To construct a conic which touches four given straight lines and passes through a given point.*

The tangent at the given point can be found by means of the reciprocal of Desargue's Theorem. [Art. 164.]

*Ex. 3. To construct a conic passing through three given points and touching two given straight lines.*

Let  $AB, AC$  be the given straight lines, and  $D, E, F$  the given points.

Let any conic through  $D, E$  touch  $AB, AC$  in  $L, M$  respectively, and let  $DE$  cut  $LM$  in the point  $P$ , and  $AB, AC$  in the points  $B', C'$  respectively.

Then, the lines  $AB, AC$ , the double line  $LMP$ , and the conic through  $D, E$  are three conics through the same four points, namely two coincident points at each of the points  $L, M$ . These conics are therefore cut in involution by the line  $DE$ , and therefore  $P$  is one of the double points of the involution determined by the pairs  $D, E$  and  $B', C'$ . Thus the chord of contact of the tangents  $AB, AC$  passes through one or other of two fixed points on  $DE$ . Similarly the chord of contact passes through one or other of two fixed points on  $DF$ .

The chord of contact is therefore one of four fixed straight lines, and if either of these lines cut  $AB, AC$  in the points  $X, Y$ , there is one and only one corresponding conic through the five points  $D, E, F, X, Y$ , which can be constructed as in Art. 157 or Art. 166.

Thus *four* conics will pass through three given points and touch two given straight lines.

Ex. 4. *To construct a conic passing through two given points and touching three given straight lines.*

Let  $ABC$  be the triangle formed by the three given tangents, and let  $D, E$  be the two given points.

Then, as in Ex. 3, the chord of contact of the tangents from  $A$  passes through one or other of two fixed points,  $S, S'$  suppose, on  $DE$ .

If  $AX$  be the polar of  $S$  with respect to the conic  $A \{SEXD\} = -1$ , and therefore  $AX$  can be constructed; and  $AX'$ , the polar of  $S'$ , can be similarly constructed. Hence  $O$ , the pole of  $DE$ , is on one or other of two fixed lines through  $A$ .

Similarly  $O$  is on one or other of two fixed lines through  $B$ .

Hence  $O$  is one of four fixed points; and when  $O$  is found,  $OD$  and  $OE$  will be the corresponding tangents to the conic, which is now completely determined since five tangents are known.

Thus *four* conics will pass through two given points and touch three given straight lines.

Ex. 5. *Construct a conic having given the poles with respect to it of three given points.*

Let  $BC, CA, AB$  be the polars of the points  $A', B', C'$  respectively.

Then, if  $K$  be the point of intersection of lines through  $B', C'$  parallel to  $AB, CA$  respectively, the centre of the conic will lie on the line  $AK$ . And, if  $L$  be the point of intersection of lines through  $A', C'$  parallel to  $AB, BC$  respectively, the centre of the conic will lie on the line  $BL$  [Art. 125]. Hence the centre,  $O$ , of the conic is found.

Let  $OA', OB', OC'$  cut  $BC, CA, AB$  in the points  $P, Q, R$  respectively. If  $A'$  and  $P$  be on the same side of  $O$ , and  $X$  be such that  $OX^2 = OA' \cdot OP$ , the line through  $X$  parallel to  $BC$  will be a tangent. Hence, if the conic be an *ellipse*, the tangents at three points can be found; and if the tangent at the point  $X$  be cut by  $OB'$  and a line

through  $O$  parallel to  $CA$  in the points  $T, T'$  respectively, the rectangle  $TX \cdot XT'$  will be equal to the square of the semi-diameter conjugate to  $OX$ . Thus a pair of conjugate diameters of the ellipse are found in position and magnitude, and the construction can be completed as in Art. 77.

If the conic be a hyperbola the asymptotes will be the double lines of the pencil in involution of which  $OA'$  and the line through  $O$  parallel to  $BC$  are one conjugate pair, and  $OB'$  and the line through  $O$  parallel to  $C'A'$  are another conjugate pair; and knowing the asymptotes and the pole of a given straight line the construction can be completed.

**171.** *The rectangle contained by the two perpendiculars drawn from any point of a conic on one pair of opposite sides of an inscribed quadrilateral, is in a constant ratio to the rectangle contained by the two perpendiculars from that point on another pair of opposite sides.* [Pappus's Theorem.]

Let  $A, B, C, D$  be four given points on a conic, and let  $O$  be any other point on the curve. Let  $Oa, Ob, Oc, Od$  be the perpendiculars from  $O$  on  $AB, BC, CD, DA$  respectively.

Then we have to prove that the ratio  $Oa \cdot Oc : Ob \cdot Od$  is constant for all positions of  $O$  on the conic.

Let  $OB, OD$  cut  $AC$  in the points  $X, Y$  respectively.

Then  $O\{ABCD\} = \{AXCY\} = AX \cdot YC : XC \cdot AY$ .

Now  $AX : XC = \triangle OAB : \triangle OBC = Oa \cdot AB : Ob \cdot BC$ ; and  $YC : AY = \triangle DCO : \triangle OAD = Oc \cdot CD : Od \cdot DA$ .

Hence, as  $O\{ABCD\}$  is constant for all positions of  $O$  on the conic, it follows that  $Oa \cdot Oc : Ob \cdot Od$  is constant.

### EXAMPLES.

1. Having given five points on a conic, shew how to find any number of other points on the curve.
2. Having given five tangents to a conic, shew how to draw any number of other tangents to the curve.
3. Construct a conic, having given the centre and three points.
4. Construct a conic, having given the centre and three tangents.

5. Construct a conic, having given two points on the curve and a triangle which is self-polar with respect to it.

6. Construct a conic, having given two tangents and a triangle which is self-polar with respect to the conic.

7. Find the centre of a rectangular hyperbola which touches four given straight lines.

8. Shew that the locus of the centres of rectangular hyperbolas with respect to which a given triangle is self-polar is the circle circumscribing the triangle.

9. Shew that, if a triangle be inscribed in the triangle  $ABC$  and touch the side  $BC$  on the point  $F$ , the centre of the conic will lie on the straight line through the middle points of  $BC$  and  $AF$ .

10.  $A, B, C, D$  are any four points on a hyperbola;  $CK$  parallel to one asymptote meets  $AD$  in  $K$  and  $DL$  parallel to the other asymptote meets  $CB$  in  $L$ . Shew that  $KL$  is parallel to  $AB$ .

11. Shew that the sixty Pascal lines corresponding to six points on a conic, intersect three by three.

12. The poles with respect to a conic of the sides  $BC, CA, AB$  are  $A', B', C'$  respectively. Shew that  $AA', BB', CC'$  meet in a point, and that the points of intersection of  $BC$  and  $B'C', CA$  and  $C'A', AB$  and  $A'B'$  are collinear.

13. Find geometrically the points where a given straight line cuts the conic determined by five given points.

14. Find geometrically the tangents from a given point to the conic determined by five given tangents.

15. Through a fixed point on a conic a line is drawn cutting the conic again in  $P$  and the sides of a given inscribed triangle in  $A', B', C'$  respectively. Shew that  $\{PA'B'C'\}$  is constant.

16. On a fixed tangent to a conic any point  $O$  is taken and  $OQ$  is the other tangent from  $O$  to the conic. Shew that, if  $ABC$  be the vertices of any triangle circumscribing the conic,  $O\{ABCQ\}$  will be constant for all positions of  $O$ .

17. Two points  $P$  and  $Q$  are conjugate with respect to a conic,  $P$  lies on a fixed straight line and  $QP$  subtends a right angle at a fixed point. Prove that the locus of  $Q$  is a conic passing through the fixed point.

18. From a fixed point  $O$  on one of the three diagonals of a complete quadrilateral tangents are drawn to the conics inscribed in the quadrilateral, shew that their points of contact lie on a conic which passes through the extremities of the two other diagonals and divides harmonically the diagonal on which  $O$  lies.

19. A straight line cuts two given circles in points which are harmonically conjugate. Shew that the line envelopes a conic whose foci are at the centres of the circles.

20. Shew how, by the method of false positions or otherwise, to describe a polygon each of whose sides shall pass through a fixed point, and each of whose vertices shall lie on a fixed straight line.

21. Shew that, if three conics pass through the same four points, a common tangent to any two of the conics is cut harmonically by the third.

22. Shew that, if three conics touch the same four straight lines, the tangents to two of the conics at a common point and the two tangents drawn from that point to the other conic form a harmonic pencil.

23. A hyperbola passes through the centre of a conic and its asymptotes are parallel to a pair of conjugate diameters of the conic; shew that an infinite number of triangles can be inscribed in the hyperbola which are self-polar with respect to the conic.

24. Shew that, if  $ABCDEF$  be any hexagon inscribed in a conic, the continued product of the perpendiculars drawn from any point on the conic to the sides  $AB$ ,  $CD$  and  $EF$  is in a constant ratio to the continued product of the perpendiculars drawn from the same point on the alternate sides  $BC$ ,  $DE$  and  $FA$ .

25. Shew that, if a conic touch the four straight lines  $AB$ ,  $BC$ ,  $CD$ ,  $DA$ , and any other tangent to the conic cut  $AD$ ,  $BC$  in  $P$ ,  $Q$  respectively; then will  $AP : PB$  and  $BQ : CQ$  be in a constant ratio. Hence, or otherwise, shew that the rectangle contained by the perpendiculars from  $A$  and  $C$  on any other tangent to the conic is in a constant ratio to the rectangle contained by the perpendicular from  $B$  and  $D$  on that tangent.

26.  $Q$  is any point on a given straight line,  $R$  is the point of intersection of the polars of  $Q$  with respect to two given conics. Shew that the locus of  $R$  for different positions of  $Q$  is a conic.

27. Any straight line is drawn through a given point  $O$ , and  $Q$ ,  $Q'$  are the poles of the line with respect to two given conics; shew that the envelope of  $QQ'$ , for different straight lines through  $O$ , is a conic.

28. From any two points  $T$ ,  $T'$  the tangents  $TP$ ,  $TQ$  and  $T'P'$ ,  $T'Q'$  are drawn to a conic; shew that the six points  $T$ ,  $T'$ ,  $P$ ,  $P'$ ,  $Q$ ,  $Q'$  lie on another conic.

29. Shew that the circum-circle of a triangle self-polar to a conic cuts orthogonally the director-circle of the conic.

30. Shew that the locus of the centres of conics inscribed in a triangle and whose director-circles have a given radius is a circle.

31. Shew that the locus of the centres of the rectangular hyperbolas which touch the sides of a triangle is the polar circle of the triangle.

32. Shew that the director-circles of all conics which touch the sides of a triangle cut orthogonally the polar circle of the triangle.

## CHAPTER VIII.

### RECIPROCATION. PROJECTION.

172. IF we have any figure consisting of any number of points and straight lines in a plane, and we take the polars of those points and the poles of the line with respect to a fixed conic  $C$ , we shall obtain another figure which is called the **polar reciprocal** of the former with respect to the **auxiliary conic  $C$** .

When a point in one figure and a line in the reciprocal figure are pole and polar with respect to the auxiliary conic  $C$ , they are said to *correspond* to one another.

If in one figure we have any curve  $S$  the lines which correspond to the different points of  $S$  will all touch some curve  $S'$ . Let the lines corresponding to the two points  $P, Q$  of  $S$  meet in  $T$ ; then  $T$  will be the pole of the line  $PQ$  with respect to the conic  $C$ , that is the line  $PQ$  corresponds to the point  $T$ . Now if the point  $Q$  move up to and ultimately coincide with  $P$ , the two corresponding tangents to  $S'$  will also ultimately coincide with one another, and their point of intersection  $T$  will ultimately be on the curve  $S'$  and will coincide with the point of contact of the line which corresponds to the point  $P$ . So that a tangent to the curve  $S$  corresponds to a point on the curve  $S'$ , just as a tangent to  $S'$  corresponds to a point on  $S$ . Thus  $S$  is generated from  $S'$  exactly as  $S'$  is from  $S$ , and we shall arrive at the same curve  $S'$  either as

the envelope of the polars of the different points on  $S$  or as the locus of the poles of the different tangents to  $S$ .

**173.** If any line  $L$  cut the curve  $S$  in any number of points  $P, Q, R, \dots$  we shall have *tangents* to  $S'$  corresponding to the *points*  $P, Q, R, \dots$ , and these tangents will all pass through a point, namely through the pole of  $L$  with respect to the auxiliary conic. Hence as many tangents to  $S'$  can be drawn through a point as there are points on  $S$  lying on a straight line; so also as many points on  $S'$  will lie on a straight line as these tangents to  $S$  drawn through a point.

The number of points, real or imaginary, in which a curve is cut by any straight line is called the **degree** of the curve, and the number of tangents which can be drawn to a curve from any point is called the **class** of the curve.

Thus the degree of a curve is equal to the class of its reciprocal, and the class of a curve is equal to the degree of its reciprocal.

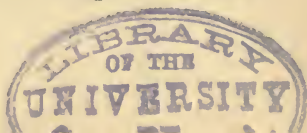
We know that a conic is of the second degree and of the second class; hence the reciprocal of a conic is also of the second degree and of the second class. We cannot however conclude from this result that the reciprocal of a conic is another conic, unless we prove that no other curves except conics are of the second degree and of the second class.

**174.** *The polar reciprocal of one conic with respect to another is a conic.*

Let  $A, B, C, D$  be any four fixed points, and let their polars with respect to the auxiliary conic be  $A'A'', B'B'', C'C'', D'D''$ .

Let  $P$  be any other point and let  $P'P''$  be its polar, and let  $P'P''$  cut  $A'A'', B'B'', C'C'', D'D''$  in the points  $a, b, c, d$  respectively.

Then, since  $a$  is the intersection of the polars of  $A$  and



$P, a$  is the pole of the line  $PA$ . Similarly  $b, c, d$  are the poles of the lines  $PB, PC, PD$  respectively.

Hence  $P\{ABCD\} = \{abcd\}$ .

Now if  $P$  be any point on a fixed conic through  $A, B, C, D$  we know that  $P\{ABCD\}$  is constant.

Hence  $\{abcd\}$  is constant, which shews that the line  $abcd$  touches a fixed conic which touches the lines  $A'A'', B'B'', C'C'', D'D''$ .

**175.** The method of Reciprocal Polars enables us to obtain from any given theorem concerning the positions of points and lines, another theorem in which straight lines take the place of points and points of straight lines.

The simplest cases of correspondence are the following:—

Points in one figure reciprocate into straight lines in the reciprocal figure.

The line joining two points reciprocate into the point of intersection of the corresponding lines.

The tangent to any curve reciprocates into a point on the reciprocal curve.

The point of contact of a tangent reciprocates into the tangent at the corresponding point.

The chord joining any two points on a curve reciprocates into the intersection of the corresponding tangents to the reciprocal curve.

The tangents to any curve from the centre of the auxiliary conic reciprocate into the points at infinity on the reciprocal curve.

The points of contact of the tangents drawn to any curve from the centre of the auxiliary conic reciprocate into the tangents at the points at infinity, that is into the asymptotes of the reciprocal curve.

[Hence, if the original curve be a conic, the line joining the points of contact of the tangents drawn from the centre of the auxiliary conic reciprocates into the centre of the reciprocal conic; also the reciprocal of a conic will be an ellipse, parabola, or hyperbola according as the tangents drawn to it from the centre of the auxiliary conic are imaginary, coincident, or real; that is according as the centre of the auxiliary conic is within, upon, or without the original conic.]

If two curves touch, that is have two coincident points common, the reciprocal curves will have two coincident tangents common, and will therefore also touch.

**176.** The following are examples of reciprocal theorems.

If the angular points of two triangles are on a conic, their six sides will touch another conic.

The three intersections of opposite sides of a hexagon inscribed in a conic lie on a straight line.

(*Pascal's Theorem.*)

If the three sides of a triangle touch a conic, and two of its angular points lie on a second conic, the locus of the third angular point is a conic.

If the sides of a triangle touch a conic, the three lines joining an angular point to the point of contact of the opposite side meet in a point.

The polars of a given point with respect to a system of conics through four given points all pass through a fixed point.

The locus of the pole of a given line with respect to a system of conics through four fixed points is a conic.

If the sides of two triangles touch a conic, their six angular points are on another conic.

The three lines joining opposite angular points of a hexagon described about a conic meet in a point. (*Brianchon's Theorem.*)

If the three angular points of a triangle lie on a conic, and two of its sides touch a second conic, the envelope of the third side is a conic.

If the angular points of a triangle lie on a conic, the three points of intersection of a side and the tangent at the opposite angular point lie on a line.

The poles of a given straight line with respect to a system of conics touching four given straight lines all lie on a fixed straight line.

The envelope of the polar of a given point with respect to a system of conics touching four fixed lines is a conic.

**177.** We now proceed to consider the results which can be obtained by reciprocating with respect to a circle.

We know that the line joining the centre of a circle to any point  $P$  is perpendicular to the polar of  $P$  with respect to the circle. Hence, if  $P, Q$  be any two points, the angle between the polars of these points with respect to a circle is equal to the angle that  $PQ$  subtends at the centre of the circle. Reciprocally the angle between any two straight lines is equal to the angle which the line joining their poles with respect to a circle subtends at the centre of the circle.

We know also that the distances, from the centre of a circle, of any point and of its polar with respect to that circle, are inversely proportional to one another.

178. If we reciprocate with respect to a circle it is clear that a change in the radius of the auxiliary circle will make no change in the *shape* of the reciprocal curve, but only in its *size*. Hence, as we are generally not concerned with the absolute magnitudes of the lines in the reciprocal figure, we only require to know the *centre* of the auxiliary circle. We may therefore speak of reciprocating with respect to a *point*  $O$ , instead of with respect to any circle having  $O$  for centre.

179. If any conic be reciprocated with respect to a point  $O$ , the points on the reciprocal figure which correspond to the tangents through  $O$  to the original curve must be at an infinite distance in directions perpendicular to those tangents. Thus the directions of the lines to the points at infinity on the reciprocal curve are perpendicular to the tangents from  $O$  to the original curve; and hence *the angle between the asymptotes of the reciprocal curve is supplementary to the angle between the tangents from  $O$  to the original curve.*

In particular, if the tangents from  $O$  to the original curve be at right angles, the reciprocal conic will be a rectangular hyperbola.

Again, the *axes* of the reciprocal conic bisect the angles between its asymptotes. The axes are therefore parallel to the bisectors of the angles between the tangents from  $O$  to the original conic.

Corresponding to the points at infinity on the original conic we have the tangents to the reciprocal conic which pass through the origin. Hence the tangents from the origin to the reciprocal conic are perpendicular to the directions of the lines to the points at infinity on the original conic, so that *the angle between the asymptotes of the original conic is supplementary to the angle between the tangents from the origin to the reciprocal conic.*

In particular, if a rectangular hyperbola be reciprocated with respect to any point  $O$ , the tangents from  $O$  to the reciprocal conic will be at right angles to one another;

in other words  $O$  is a point on the director-circle of the reciprocal conic.

**180.** Let any line through  $O$  cut a given conic in the points  $P, P'$ , and let the tangents at  $P, P'$  meet in  $T$ ; then we know that  $T$  is on the polar of  $O$  with respect to the given conic.

If now we reciprocate with respect to  $O$ :—corresponding to the points  $P, P'$  on a line through  $O$  are two *parallel* tangents to the reciprocal conic, and corresponding to the intersection of the tangents to the original conic at  $P, P'$  is the line joining the points of contact of these parallel tangents. Hence corresponding to the line on which  $T$  lies is the point through which the chord of contact of parallel tangents to the reciprocal conic passes, and this point is the centre of the reciprocal conic.

Thus *the polar of the origin with respect to any conic reciprocates into the centre of the reciprocal conic.*

**181.** The following are important examples of reciprocation :

We know that *all conics which circumscribe a triangle and pass through its orthocentre are rectangular hyperbolas.*

Reciprocating with respect to the orthocentre  $O$  we shall obtain another triangle with the same orthocentre.

The rectangular hyperbolas will become parabolas, since they all pass through  $O$ ; and, since the points at infinity on any one of the conics are in perpendicular directions, the tangents from  $O$  to any one of the parabolas will be at right angles, so that the point  $O$  is on the directrix of each parabola.

Thus the reciprocal theorem is *the directrices of all parabolas which touch the three sides of a given triangle pass through the orthocentre of the triangle.*

Again, the original theorem may be expressed in the form ;—



Upon  $SP$  take the point  $Y$  such that  $SP \cdot SY$  may be equal to the square of the radius of the auxiliary circle. Then the line through  $Y$  perpendicular to  $SP$  will be the polar of  $P$  with respect to the auxiliary circle, and will therefore be a tangent to the reciprocal curve.

Since the rectangles  $SP \cdot SY$  and  $SP \cdot SP'$  are both constant, it follows that the ratio  $SY : SP'$  is constant, and therefore the locus of  $Y$  is similar to the locus of  $P'$ ; the locus of  $Y$  for different positions of  $P$  is therefore a circle. [If the point  $S$  be on the circumference of the given circle, the locus of  $Y$  will be a straight line, that is a circle of infinite radius.]

But a straight line which moves so that the locus of the foot of the perpendicular upon it, drawn from a fixed point, is a circle, envelopes a conic of which the fixed point is a focus.

Hence *the polar reciprocal of a circle with respect to any point  $S$  is a conic of which  $S$  is a focus.*

Draw  $YC$  parallel to  $OP'$  and cutting  $OS$  in  $Y$ .

Then, from similar triangles,

$$SC : SO = SY : SP' = \text{constant},$$

and  $CY : CS = OP' : OS = \text{constant}.$

Thus  $C$  is the *centre* of the reciprocal conic, and  $CY$  is the length of its semi-major-axis.

Hence the *eccentricity* of the conics is equal to

$$CS : CY = OS : OP'.$$

The *reciprocal conic is therefore an ellipse, a parabola, or a hyperbola according as the point  $S$  is within, upon, or without the given circle.*

Let  $OD$  be the radius of the circle which is perpendicular to  $OS$ . Draw  $SN$ , the perpendicular on the tangent at  $D$ , and take a point  $L$  on  $SN$  such that  $SN \cdot SL$  may be equal to the square of the radius of the auxiliary circle. Then  $L$  will be the point on the reciprocal conic which corresponds to the tangent  $DN$ , and  $SL$  will be the

semi-latus-rectum of the conic. And, since  $SN = OD$ , it follows that *the diameter of the auxiliary circle is a mean proportional to the diameter of the given circle and the latus-rectum of the reciprocal conic.*

Again, since  $L$  is the polar of  $DN$  with respect to the auxiliary circle, the polar of  $D$  will pass through  $L$ , and it will be perpendicular to  $SD$ ; hence, as this polar touches the reciprocal conic, it must be the tangent at the point  $L$ . Thus the tangent at  $L$  is the line through  $L$  perpendicular to  $SD$ ; and this tangent will cut  $SO$  in a point  $X$  which is the foot of the directrix corresponding to the focus  $S$ .

Now, since the angles  $DOX$  and  $DFX$  are right angles,

$$SO \cdot SX = SD \cdot SF$$

= square of radius of auxiliary circle,

whence it follows that *the centre  $O$  of the given circle reciprocates into the directrix of the conic.*

If the polar of  $S$  with respect to the given circle cut  $OS$  in the point  $K$ ,  $OS \cdot OK = OP^2$ , and therefore

$$OS \cdot SK = OP^2 - OS^2 = PS \cdot SP'.$$

Hence

$$SK : SP = SP' : OS$$

$$= SY : SC;$$

$$\therefore SK \cdot SC = SP \cdot SY,$$

whence it follows that *the polar of  $S$  with respect to the given circle reciprocates into the centre of the reciprocal conic.* [See Art. 180.]

Ex. 1. *Tangents to a conic subtend equal angles at a focus.*

Reciprocate with respect to the focus :—then corresponding to the two tangents to the conic, there are two points on a circle; the point of intersection of the tangents to the conic corresponds to the line joining the two points on the circle; and the points of contact of the tangents to the conic correspond to the tangents at the points on the circle. Also the angle subtended at the focus of the conic by any two points is equal to the angle between the lines corresponding to

those two points. Hence the reciprocal theorem is—The line joining two points on a circle makes equal angles with the tangents at those points.

Ex. 2. *The envelope of the chord of a conic which subtends a right angle at a fixed point  $O$  is a conic having  $O$  for a focus, and the polar of  $O$ , with respect to the original conic, for the corresponding directrix.*

Reciprocate with respect to  $O$ , and the proposition becomes—The locus of the point of intersection of tangents to a conic which are at right angles to one another is a concentric circle.

Ex. 3. *If two conics have a common focus, two of their common chords will pass through the intersection of their directrices.*

Reciprocate with respect to the common focus, and the proposition becomes—Two of the points of intersection of the common tangents to two circles are on the line joining the centres of the circles.

Ex. 4. *The orthocentre of a triangle circumscribing a parabola is on the directrix.*

Reciprocating with respect to the orthocentre we obtain—A conic circumscribing a triangle and passing through the orthocentre is a rectangular hyperbola.

**184.** *To reciprocate a system of circles with the same radical axis into a system of confocal conics.*

If we reciprocate with respect to *any* point  $O$  we obtain a system of conics having  $O$  for one focus, and [Art. 183] the centre of any conic is the reciprocal of the polar of  $O$  with respect to the corresponding circle. Now either of the two 'limiting points' of the system is such that its polar with respect to any circle of the system is a fixed straight line, namely a line through the other limiting point parallel to the radical axis. If therefore the system of circles be reciprocated with respect to a limiting point the reciprocals will have the same centre; and if they have a common centre and one common focus they will be confocal. Since the radical axis is parallel to and midway between a limiting point and its polar, the reciprocal of the radical axis (with respect to the limiting point) is on the line through the focus and centre of the reciprocal conics, and is twice as far from the focus as the

centre; so that when we reciprocate a system of coaxial circles with respect to a limiting point, the radical axis reciprocates into the other focus of the system of confocal conics.

The following theorems are reciprocal :

The tangents at a common point of two confocal conics are at right angles.

The locus of the point of intersection of two lines, each of which touches one of two confocal conics, and which are at right angles to one another, is a circle.

If from any point two pairs of tangents  $P, P'$  and  $Q, Q'$  be drawn to two confocal conics; the angle between  $P$  and  $Q$  is equal to that between  $P'$  and  $Q'$ .

The points of contact of a common tangent to two circles subtend a right angle at one of the limiting points.

The envelope of the line joining two points, each of which is on one of two circles, and which subtend a right angle at a limiting point, is a conic one of whose foci is at the limiting point.

If any straight line cut two circles in the points  $P, P'$  and  $Q, Q'$ ; the angles subtended at a limiting point by  $PQ$  and  $P'Q'$  are equal.

## CONICAL PROJECTION.

**185.** If any point  $P$  be joined to a fixed point  $V$ , and  $VP$  be cut by any fixed plane in  $P'$ , the point  $P'$  is called the *projection* of  $P$  on that plane. The point  $V$  is called the *vertex* or the *centre of projection*, and the cutting plane is called the *plane of projection*.

**186.** *The projection of any straight line is a straight line.*

For the straight lines joining  $V$  to all the points of any straight line are in a plane, and this is cut by the plane of projection in a straight line.

**187.** *Any plane curve is projected into a curve of the same degree.*

For, if any straight line meet the original curve in any number of points  $A, B, C, D, \dots$ , the projection of the line will meet the projection of the curve where  $VA, VB,$

$VC$ ,  $VD$ ... meet the plane of projection. There will therefore be the *same number* of points on a straight line in the one curve as in the other. This proves the proposition.

**188.** *A tangent to a curve projects into a tangent to the projected curve.*

For, if a straight line meet a curve in two points  $A$ ,  $B$ , the projection of that line will meet the projected curve in two points  $a$ ,  $b$  where  $VA$ ,  $VB$  meet the plane of projection. Now if  $A$  and  $B$  coincide, so also will  $a$  and  $b$ .

**189.** *The relation of pole and polar with respect to a conic are unaltered by projection.*

This follows from the two preceding Articles.

It is also clear that two conjugate points, or two conjugate lines, with respect to a conic, project into conjugate points, or lines, with respect to the projected conic.

**190.** Draw through the vertex a plane parallel to the plane of projection, and let it cut the original plane in the line  $K'L'$ . Then, since the plane  $VK'L'$  and the plane of projection are parallel, their line of intersection, which is the projection of  $K'L'$ , is at an infinite distance.

Hence to project any particular straight line  $K'L'$  to an infinite distance, take any point  $V$  for vertex and a plane parallel to the plane  $VK'L'$  for the plane of projection.

Straight lines which meet in any point on the line  $K'L'$  will be projected into parallel straight lines, for their point of intersection will be projected to infinity.

**191.** *A system of parallel lines on the original plane will be projected into lines which meet in a point.*

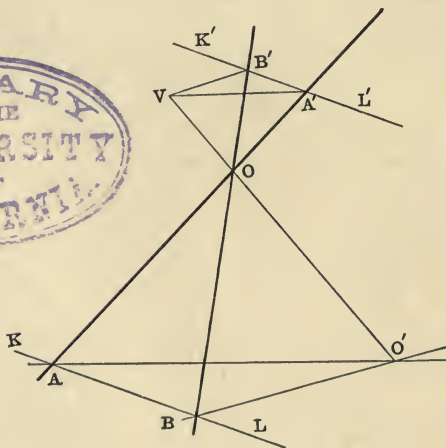
For, let  $VP$  be the line through the vertex parallel to the system,  $P$  being on the plane of projection; then,

since  $VP$  is in the plane through  $V$  and any one of the parallel lines, the projection of every one of the parallel lines will pass through  $P$ .

For different systems of parallel lines the point  $P$  will change; but, since  $VP$  is always parallel to the original plane, the point  $P$  is always on the line of intersection of the plane of projection and a plane through the vertex parallel to the original plane.

Hence any system of parallel lines on the original plane is projected into a system of lines passing through a point, and all such points, for different systems of parallel lines, are on a straight line.

**192.** Let  $KL$  be the line of intersection of the original plane and the plane of projection. Draw through the vertex a plane parallel to the plane of projection, and let it cut the original plane in the line  $K'L'$ . Let the two straight lines  $AOA'$ ,  $BOB'$  meet the lines  $KL$ ,  $K'L'$  in the points  $A$ ,  $B$  and  $A'$ ,  $B'$  respectively; and let  $VO$  meet the plane of projection in  $O'$ . Then  $AO'$  and  $BO'$  are the projections of  $AOA'$  and  $BOB'$ .



Since the planes  $VA'B'$ ,  $AO'B$  are parallel, and parallel planes are cut by the same plane in parallel lines, the lines  $VA'$ ,  $VB'$  are parallel respectively to  $AO'$ ,  $BO'$ . The angle  $A'VB'$  is therefore equal to the angle  $AO'B$ , that is,  $A'VB'$  is equal to the angle into which  $AOB$  is projected.

Similarly, if the straight lines  $CD$ ,  $ED$ , meet  $K'L'$  in  $C'$ ,  $D'$  respectively, the angle  $C'VD'$  will be equal to the angle into which  $CDE$  is projected.

From the above we obtain the fundamental proposition in the theory of projections, viz.,

*Any straight line can be projected to infinity, and at the same time any two angles into given angles.*

For, let the straight lines bounding the two angles meet the line which is to be projected to infinity in the points  $A'$ ,  $B'$  and  $C'$ ,  $D'$ ; draw any plane through  $A'B'C'D'$ , and in that plane draw segments of circles through  $A'$ ,  $B'$  and  $C'$ ,  $D'$  respectively containing angles equal to the two given angles. Either of the points of intersection of these segments of circles may be taken for the centre of projection, and the plane of projection must be taken parallel to the plane we have drawn through  $A'B'C'D'$ .

If the segments do not meet, the centre of projection is imaginary.

**Ex. 1.** *To shew that any quadrilateral can be projected into a square.*

Let  $ABCD$  be the quadrilateral; and let  $P$ ,  $Q$  [see figure p. 208] be the points of intersection of a pair of opposite sides, and let the diagonals  $BD$ ,  $AC$  meet the line  $PQ$  in the points  $S$ ,  $R$ . Then, if we project  $PQ$  to infinity and at the same time the angles  $PDQ$  and  $ROS$  into right angles, the projection must be a square. For, since  $PQ$  is projected to infinity, the pairs of opposite sides of the projection will be parallel, that is to say, the projection is a parallelogram; also one of the angles of the parallelogram is a right angle, and the angle between the diagonals is a right angle; hence the projection is a square.

**Ex. 2.** *To shew that the triangle formed by the diagonals of a quadrilateral is self-polar with respect to any conic which touches the sides of the quadrilateral.*

Project the quadrilateral into a square; then, the circle circumscribing the square is the director-circle of the conic, therefore

the intersection of the diagonals of the square is the centre of the conic.

Now the polar of the centre is the line at infinity; hence the polar of the point of intersection of two of the diagonals is the third diagonal.

Ex. 3. *If a conic be inscribed in a quadrilateral the line joining two of the points of contact will pass through one of the angular points of the triangle formed by the diagonals of the quadrilateral.*

Ex. 4. *If  $ABC$  be a triangle circumscribing a parabola, and the parallelograms  $ABA'C$ ,  $BCB'A$ , and  $CAC'B$  be completed; then the chords of contact will pass respectively through  $A'$ ,  $B'$ ,  $C'$ .*

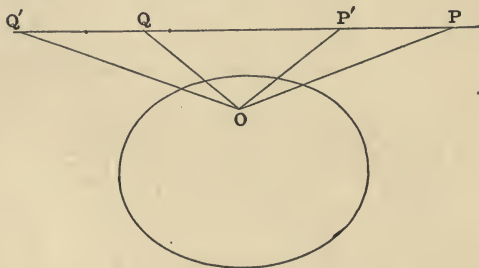
This is a particular case of Ex. 3, one side of the quadrilateral being the line at infinity.

Ex. 5. *If the three lines joining the angular points of two triangles meet in a point, the three points of intersection of corresponding sides will lie on a straight line.*

Project two of the points of intersection of corresponding sides to infinity, then two pairs of corresponding sides will be parallel, and it is easy to shew that the third pair will also be parallel.

Ex. 6. *Any two conics can be projected into concentric conics.* [See Art. 162, Ex. 2.]

**193.** *Any conic can be projected into a circle having the projection of any given point for centre.*



Let  $O$  be the point whose projection is to be the centre of the projected curve.

Let  $P$  be any point on the polar of  $O$ , and let  $OQ$  be the polar of  $P$ ; then  $OP$  and  $OQ$  are conjugate lines.

Take  $OP'$ ,  $OQ'$  another pair of conjugate lines.

Then project the polar of  $O$  to infinity, and the angles

$POQ$ ,  $P'OQ'$  into right angles. We shall then have a conic whose centre is the projection of  $O$ , and since two pairs of conjugate diameters are at right angles, the conic is a circle.

**194.** *A system of conics inscribed in a quadrilateral can be projected into confocal conics.*

Let two of the sides of the quadrilateral intersect in the point  $A$ , and the other two in the point  $B$ . Draw any conic through the points  $A$ ,  $B$ , and project this conic into a circle, the line  $AB$  being projected to infinity; then,  $A$ ,  $B$  are projected into the circular points at infinity, and since the tangents from the circular points at infinity to all the conics of the system are the same, the conics must be confocal.

**Ex. 1.** *Conics through four given points can be projected into coaxial circles.*

For, project the line joining two of the points to infinity, and one of the conics into a circle; then all the conics will be projected into circles, for they all go through the circular points at infinity.

**Ex. 2.** *Conics which have double contact with one another can be projected into concentric circles.*

**Ex. 3.** *The three points of intersection of opposite sides of a hexagon inscribed in a conic lie on a straight line. [Pascal's Theorem.]*

Project the conic into a circle, and the line joining the points of intersection of two pairs of opposite sides to infinity; then we have to prove that if two pairs of opposite sides of a hexagon inscribed in a circle are parallel, the third pair are also parallel.

**Ex. 4.** *Shew that all conics through four fixed points can be projected into rectangular hyperbolas.*

There are three pairs of lines through the four points, and if two of the angles between these pairs of lines be projected into right angles, all the conics will be projected into rectangular hyperbolas. [Art. 121.]

**Ex. 5.** *If two triangles are self-polar with respect to a conic, their six angular points are on a conic, and their six sides touch a conic.*

Let the triangles be  $ABC$ ,  $A'B'C'$ . Project  $BC$  to infinity, and the conic into a circle; then  $A$  is projected into the centre of the

circle, and  $AB$ ,  $AC$  are at right angles, since  $ABC$  is self-polar; also, since  $A'B'C'$  is self-polar with respect to the circle,  $A$  is the ortho-centre of the triangle  $A'B'C'$ .

Now a rectangular hyperbola through  $A'$ ,  $B'$ ,  $C'$  will pass through  $A$ , and a rectangular hyperbola through  $B$  will go through  $C$ . Hence, since a rectangular hyperbola can be drawn through any four points, the six points  $A$ ,  $B$ ,  $C$ ,  $A'$ ,  $B'$ ,  $C'$  are on a conic.

Also a parabola can be drawn to touch the four straight lines  $B'C'$ ,  $C'A'$ ,  $A'B'$ ,  $AB$ . And  $A$  is on the directrix of the parabola [Art. 53, Ex. 4]; therefore  $AC$  is a tangent. Hence a conic touches the six sides of the two triangles.

**195.** Properties of a figure which are true for any projection of that figure are called *projective properties*. In general such properties do not involve magnitudes. There are however some projective properties in which the magnitudes of lines and angles are involved: the most important of these is the following:—

*The cross-ratios of pencils and ranges are unaltered by projection.*

Let  $A$ ,  $B$ ,  $C$ ,  $D$  be four points in a straight line, and  $A'$ ,  $B'$ ,  $C'$ ,  $D'$  be their projections. Then, if  $V$  be the centre of projection,  $VAA'$ ,  $VBB'$ ,  $VCC'$ ,  $VDD'$  are straight lines; and we have [Art. 151]

$$\{ABCD\} = V\{ABCD\} = \{A'B'C'D'\}.$$

If we have any pencil of four straight lines meeting in  $O$ , and these be cut by any transversal in  $A$ ,  $B$ ,  $C$ ,  $D$ ; then

$$\begin{aligned} O\{ABCD\} &= \{ABCD\} = V\{ABCD\} = \{A'B'C'D'\} \\ &= O'\{A'B'C'D'\}. \end{aligned}$$

From the above together with Article 154 it follows that *if any number of points be in involution, their projections will be in involution.*

**Ex. 1.** Any chord of a conic through a given point  $O$  is divided harmonically by the curve and the polar of  $O$ .

Project the polar of  $O$  to infinity, then  $O$  is the centre of the projection, the chord therefore is bisected in  $O$ , and  $\{POQ\infty\}$  is harmonic when  $PO = OQ$ .

Ex. 2. *Conics through four fixed points are cut by any straight line in pairs of points in involution.* [Desargue's Theorem.]

Project two of the points into the circular points at infinity, then the conics are projected into coaxial circles, and the proposition is obvious.

Ex. 3. *If  $AOA'$ ,  $BOB'$ ,  $COC'$ ,  $DOD'$ ,... be chords of a conic, the points  $A, B, C, D, \dots$  and the points  $A', B', C', D', \dots$  will subtend homographic pencils at any point on the curve.*

Project the conic into a circle having  $O$  for centre.

Ex. 4. *If there are two systems of points on a conic which subtend homographic pencils at any point on the curve, the lines joining corresponding points of the two systems will envelope a conic having double contact with the original conic.*

Let  $A, B, C, D, \dots$  and  $A', B', C', D', \dots$  be the two systems of points.

Let  $AB', A'B$  meet in  $K$ , and  $AC', A'C$  in  $L$ ; and project the conic into a circle,  $KL$  being projected to infinity. Then, it is easily seen that  $AA', BB'$  and  $CC'$  will be projected into equal chords of the circle; and therefore the angles subtended by  $AB$  and  $BC$  at any point on the circle will be equal respectively to the angles subtended by  $A'B'$  and  $B'C'$ .

Hence, if  $P, P'$  be any other pair of corresponding points, and  $O$  be any point on the circle, since  $O\{ABCP\} = O\{A'B'C'P'\}$ , it follows that the angles  $COP$  and  $C'OP'$  are equal, and that

$$PP' = CC' = BB' = AA'.$$

The envelope of  $PP'$  is therefore a concentric circle.

Ex. 5. *In a given conic inscribe a triangle each of whose sides will pass through a given fixed point.*

Let  $P, Q, R$  be the three fixed points through which the sides pass.

Draw any chord  $B'C'$  through the point  $P$ , and let  $C'Q$  cut the curve again in  $A'$ , and let  $A'R$  cut the curve again in  $D'$ .

Let any other points  $B'', B'''$  be taken on the conic, and the corresponding points  $D'', D'''$  be found.

Let  $X$  be either of the points on the conic which are such that

$$O\{B'B''B'''X\} = O\{D'D''D'''X\},$$

where  $O$  is any point on the conic [see Art. 168, Ex. 4]. Then, if  $XP$  cut the curve again in  $Y$ , and  $YQ$  cut the curve in  $Z$ ,  $ZX$  will pass through  $R$ , and  $XYZ$  will be one of the two real or imaginary triangles which satisfy the required conditions.

**196.** The following are additional examples of the methods of reciprocation and projection.

*Ex. 1. If the sides of a triangle touch a conic, and if two of the angular points move on fixed confocal conics, the third angular point will describe a confocal conic.*

Let  $ABC, A'B'C'$  be two indefinitely near positions of the triangle, and let  $AA', BB', CC'$  produced form the triangle  $PQR$ . The six points  $A, B, C, A', B', C'$  are on a conic [Art. 167, Ex. 1], and this conic will ultimately touch the sides of  $PQR$  in the points  $A, B, C$ . Hence  $PA, QB, RC$  will meet in a point; and it is easily seen that the pencils  $A\{QCPB\}, B\{RAQC\}, C\{PBRA\}$  are harmonic. Now, if  $A$  move on a conic confocal to that which  $AB, AC$  touch, the tangent at  $A$ , that is the line  $QR$ , will make equal angles with  $AB, AC$ . Hence, since  $A\{QCPB\}$  is harmonic,  $PA$  is perpendicular to  $QR$ . Similarly, if  $B$  move on a confocal,  $QB$  is perpendicular to  $RP$ . Hence  $RC$  must be perpendicular to  $PQ$ , and therefore  $CA, CB$  make equal angles with  $PQ$ ; whence it follows that  $C$  moves on a confocal conic.

[The proposition can easily be extended. For, let  $ABCD$  be a quadrilateral circumscribing a conic, and let  $A, B, C$  move on confocals. Let  $DA, CB$  meet in  $E$ , and  $AB, DC$  in  $F$ . Then, by considering the triangles  $ABE, BCF$ , we see that  $E$  and  $F$  move on confocals. Hence, by considering the triangle  $CED$ , we see that  $D$  will move on a confocal.]

If we reciprocate with respect to a focus we obtain the following theorem:

*If the angular points of a triangle are on a circle of a co-axial system, and two of the sides touch circles of the system, the third side will touch another circle of the system. [Poncelet's theorem.]*

**Ex. 2.** *The six lines joining the angular points of a triangle to the points where the opposite sides are cut by a conic, will touch another conic.*

The reciprocal theorem is:—

*The six points of intersection of the sides of a triangle with the tangents to a conic drawn from the opposite angular points, will lie on another conic.*

Project two of the points into the circular points at infinity, then the opposite angular point of the triangle will be projected into a focus, and we have the obvious theorem:—

*Two lines through a focus of a conic are cut by pairs of tangents parallel to them in four points on a circle.*

**Ex. 3.** The following theorems are deducible from one another.

(i) *Two lines at right angles to one another are tangents one to each of two confocal conics; shew that the locus of their intersection is a circle, and that the envelope of the line joining their points of contact is another confocal.*

(ii) Two points, one on each of two co-axial circles, subtend a right angle at a limiting point; shew that the envelope of the line joining them is a conic with one focus at the limiting point, and that the locus of the intersection of the tangents at the points is a co-axial circle.

(iii) Two lines which are tangents one to each of two conics, cut a diagonal of their circumscribing quadrilateral harmonically; shew that the locus of the intersection of the lines is a conic through the extremities of that diagonal, and that the envelope of the line joining the points of contact is a conic inscribed in the same quadrilateral.

(iv)  $AOB$ ,  $COD$  are common chords of two conics, and  $P$ ,  $Q$  are points, one on each conic, such that  $O\{APBQ\}$  is harmonic; shew that the envelope of the line  $PQ$  is a conic touching  $AB$ ,  $CD$ , and that the tangents at  $P$ ,  $Q$  meet on a conic through  $A$ ,  $B$ ,  $C$ ,  $D$ .

(v) If two points be taken, one on each of two circles, equidistant from their radical axis, the envelope of the line joining them is a parabola which touches the radical axis, and the locus of the intersection of the tangents at the points is a circle through their common points.

# EXAMPLES.

1. Shew that four conics can be described having a common focus and passing through three given points, and that the latus-rectum of one of the conics is equal to the sum of the latera recta of the other three.

2. Reciprocate with respect to any point the theorem:—‘The tangents to a circle from any point make equal angles with their chord of contact.’

3. Reciprocate the following:—‘The common chord of two circles is perpendicular to the line joining the centres of the circles.’

4. Reciprocate the following:—‘Four circles will touch three given straight lines, and the line joining the centres of any two of the circles will pass through one of the points of intersection of the given straight lines.’

5. Shew that, if a triangle be reciprocated with respect to any point  $O$  on its circum-circle, the point  $O$  will be on the circum-circle of the reciprocal triangle.

6. Two parabolas touch the sides of a given triangle and cut one another at right angles at  $P$ . Shew that the point  $P$  must lie on the circum-circle of the given triangle.

7. Reciprocate with respect to  $B$  the following theorem:—‘If any one of a system of circles through the two fixed points  $A$ ,  $B$  be cut by any given straight line through  $A$  in the point  $P$ , the tangent at  $P$  will, for different circles of the system, envelope a parabola whose focus is  $B$ .’

8. Reciprocate with respect to one of the limiting points the theorem:—‘The tangents drawn to all the circles of a coaxial system from any point on the radical axis are equal.’

9. Reciprocate with respect to  $O$  and  $O'$  the theorem:—‘The locus of the centre of a circle which passes through a fixed point  $O$  and touches a fixed circle whose centre is  $O'$  is a conic whose foci are  $O$  and  $O'$ .’

10. Reciprocate and so prove the following theorem:—‘Two chords of a rectangular hyperbola are at right angles to one another, and each subtends a right angle at a fixed point  $P$ .’ Shew that the locus of their intersection is the polar of  $P$ .

11. Any line is drawn through a fixed point  $O$  cutting a given conic in the points  $P$ ,  $Q$ , and the point  $R$  is taken on the line such that  $\{OPQR\}$  is constant. Shew that the locus of  $R$  is a conic having double contact with the given conic.

12. Shew that two circles and their centres of similitude subtend a pencil in involution at any point.

13. Shew that, if  $P$ ,  $P'$  be corresponding points of two homographic ranges on the fixed lines  $OA$ ,  $OA'$  respectively, and the parallelogram  $POP'Q$  be completed, the locus of  $Q$  will be a conic.

14. Shew that any conic which passes through the three fixed points  $A$ ,  $B$ ,  $C$ , and is such that two other given points are conjugate with respect to it, will pass through another fixed point.

15. Shew that the locus of the centre of a conic which passes through the two fixed points  $A$ ,  $B$ , and has also two given pairs of conjugate points, is a conic.

16. A conic circumscribes a triangle and its director-circle passes through the orthocentre of the triangle; shew that the polar of the orthocentre with respect to the conic touches the polar-circle of the triangle.

17. Prove that, if a conic be drawn through the four points of intersection of two given conics, and through the intersection of one pair of common tangents, it also passes through the intersection of the other pair of common tangents.

18. Prove that the locus of the vertex from which a system of four fixed points in a plane can be projected into a square is a circle in a plane at right angles to the third diagonal of the quadrilateral formed by the four points.

19. Shew that any two triangles in plane perspective can be projected into equilateral triangles.

20. A circle and a rectangular hyperbola are described each with its centre on the other curve; a parabola is described with its focus at the centre of the hyperbola and its directrix touching the hyperbola at the centre of the circle; prove that there are an infinite number of triangles, which are at the same time inscribed in one of the three curves, circumscribed about another and self-polar with respect to the third, in any order.

21. Shew that the envelope of the axes of conics which touch two given straight lines at fixed points is a parabola.

22. Shew that if one conic inscribed in a quadrilateral is a circle, the axes of any other inscribed conic are the asymptotes of a rectangular hyperbola which circumscribes the diagonal triangle of the quadrilateral; shew also that the axes envelope a parabola which touches the diagonals of the quadrilateral and whose directrix passes through the middle points of the diagonals.



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